Subanalytic Stratifications and Bisimulations*

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Abstract. Decidability results for the verification of hybrid systems consist of constructing special finite state quotients called bisimulations whose properties are equivalent to those of the original infinite state system. This approach has had success in the case of timed automata and linear hybrid automata. In this paper, the powerful frameworks of stratification theory and subanalytic sets are presented and used in order to obtain bisimulations of certain analytic vector fields on analytic manifolds.

1 Introduction

Hybrid systems consist of finite state machines interacting with differential equations. Various modeling formalisms, analysis, design and control methodologies, as well as applications, can be found in [2–4, 10, 13]. The theory of formal verification is one of the main approaches for analyzing properties of hybrid systems. The system to be analyzed is first modeled as a hybrid automaton, and the property to be analyzed is expressed using a formula from some temporal logic. Then, model checking or deductive algorithms are used in order to guarantee that the system model indeed satisfies the desired property.

Many verification algorithms are essentially reachability algorithms which check whether the system can reach certain undesirable regions of the state space. Even though for finite state, discrete systems this approach has had success, when dealing with the infinite state space of a hybrid automaton, model checking algorithms are in danger of not terminating. Decidability results for analyzing hybrid systems consider special finite state quotients of the original infinite state hybrid automaton called bisimulations [11]. Bisimulations are special quotient systems in the sense that checking a property on the quotient system is *equivalent* to checking the property on the original system. If an infinite state hybrid automaton has a finite state bisimulation then the analysis and verification procedure is decidable.

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Obtaining bisimulations for purely discrete, finite state automata is clearly decidable since the underlying state space is finite. Correspondingly, the process of constructing bisimulations for hybrid systems may not terminate because of the infinite cardinality of the continuous state space and dynamics. In this paper, we consider the problem of constructing finite state bisimulations for purely continuous systems. More precisely, *given an analytic vector field on an analytic manifold, a set of initial conditions and a set of unsafe states, we would like to construct a finite state transition system such that checking reachability on the finite graph is equivalent to checking reachability of the original continuous system.*

In order to tackle this problem, the powerful frameworks of subanalytic sets and stratification theory [5, 12, 16] are used. Subanalytic sets are an important class of sets having many desirable "finiteness" properties. For example, relatively compact subanalytic sets have finitely many connected components. In addition, subanalytic sets are closed under intersections, unions, complementation as well as forward images under proper maps and inverse images. Stratification theory allows us to deal with many technical issues concerning sets and their boundaries and is crucial in refining partitions. With these tools we present an algorithm for constructing bisimulations of analytic systems as well as a proof that the algorithm terminates in the case of linear vector fields in $\mathbb{R}^2$ with real or purely imaginary eigenvalues.

The outline of the paper is as follows: In Section 2 we review the notion of bisimulations as well as the algorithm for computing bisimulations for transition systems. Section 3 presents some basic facts about stratification theory and subanalytic sets and in Section 4 we use these facts to construct bisimulations of analytic vector fields. Finally, Section 5 presents interesting issues for further research.

2 Bisimulations

A more detailed exposition of the material described in this section can be found in [11]. A transition system $H = (Q, \Sigma, \rightarrow, Q_O, Q_F)$ consists of a set Q of states, an alphabet Σ of events, a transition relation $\rightarrow \subseteq Q \times \Sigma \times Q$, a set $Q_O \subseteq Q$ of initial states, and a set $Q_F \subseteq Q$ of final states. The transition system is finite if the cardinality of Q is finite and it is infinite otherwise. A region is a subset $R \subseteq Q$. Given $\sigma \in \Sigma$ we define $Pre_\sigma(R)$ as

$$Pre_\sigma(R) = \{q \in Q \mid \exists p \in R \text{ and } (q, \sigma, p) \in \rightarrow\}$$

and $Pre(R)$ as

$$Pre(R) = \bigcup_{\sigma \in \Sigma} Pre_\sigma(R)$$

Let $\sim \subseteq Q \times Q$ be an equivalence relation on the state space and let $Q/\sim$ denote the resulting quotient space. A $\sim$-block is a union of equivalence classes. For a region R we denote by $R/\sim$ the smallest $\sim$-block that contains R. Thus,

$Q_O/\sim$ and $Q_F/\sim$ are $\sim$-blocks containing the initial and final states respectively. The transition relation $\rightarrow_\sim$ on the quotient space is defined as follows: for Q_1,$Q_2 \in Q/\sim$, $(Q_1, \sigma, Q_2) \in \rightarrow_\sim$ iff there exist $q_1 \in Q_1$ and $q_2 \in Q_2$ such that $(q_1, \sigma, q_2) \in \rightarrow$. The quotient transition system is then $H/\sim = (Q/\sim, \Sigma, \rightarrow_\sim, Q_0/\sim, Q_F/\sim)$.

The quotient system $H/\sim$ is a *bisimulation* of H iff Q_F is a $\sim$-block and for all $\sigma \in \Sigma$ and all $\sim$-blocks R, the region $Pre_\sigma(R)$ is a $\sim$-block. A bisimulation is called finite if it has a finite number of equivalence classes. Bisimulations are very important because bisimilar transition systems generate the same language. Therefore, checking properties on the bisimilar quotient is equivalent to checking properties of the original transition system. This is very useful in reducing the complexity of various verification algorithms. In addition, if H is infinite and $H/\sim$ is a finite bisimulation, then verification algorithms for infinite systems (for example, hybrid systems) are guaranteed to terminate. A successful application of this approach for timed automata can be found in [1].

Two states p,$q \in Q$ are bisimilar denoted $p \approx q$ iff there exists a bisimulation $\sim$ such that $p \sim q$. It can be shown that if $p \approx q$ then

1. $p \in Q_F$ iff $q \in Q_F$
2. if $(p, \sigma, p') \in \rightarrow$ then there exists q' such that $(q, \sigma, q') \in \rightarrow$ and $p' \approx q'$
3. if $(q, \sigma, q') \in \rightarrow$ then there exists p' such that $(p, \sigma, p') \in \rightarrow$ and $p' \approx q'$

It should be noted that the notion of bisimulation is very similar to the notion of dynamic consistency [7]. Given a transition system H, the following algorithm computes the bisimilarity partition. The algorithm terminates if the bisimilarity quotient is finite.

Algorithm (Bisimilarity for transition systems)
Set $Q/\sim = \{Q_F, Q \setminus Q_F\}$
while $\exists$ R,$R' \in Q/\sim$ and $\sigma \in \Sigma$ such that $\emptyset \subset R \cap Pre_\sigma(R') \subset R$, **do**
  refine $Q/\sim = (Q/\sim \setminus \{R\}) \cup \{R \cap Pre_\sigma(R'), R \setminus Pre_\sigma(R')\}$
end while

Initially the quotient space consists of two equivalence classes, Q_F and $Q \setminus Q_F$ (here $\setminus$ denotes set difference). The algorithm then checks whether there exist $\sim$-equivalence classes whose preimage under Pre_σ for some σ is neither empty nor a $\sim$-equivalence class. If there are none then a bisimilarity quotient has been reached. Otherwise there exists R, $R' \in Q/\sim$ such that $R \cap Pre_\sigma(R') \neq \emptyset$ and $R \cap Pre_\sigma(R')$ is a proper subset of R for some $\sigma \in \Sigma$. Then the algorithm refines the partition by splitting R into $R \cap Pre_\sigma(R')$ and $R \setminus Pre_\sigma(R')$. This procedure is repeated either forever or until a bisimilarity quotient is reached.

Inspired by the above bisimulation algorithm, we would like to have an algorithm for obtaining *finite* bisimulations of analytic vector fields. More precisely, the original transition system consists of a (infinite cardinality) real analytic manifold M and the transition relation is generated by the flow of an analytic vector field. A collection of subsets $\mathcal{A}$ of M can be used to describe initial conditions, guards conditions, invariants as well as undesirable regions of the state space. These sets typically exist within each discrete location of a hybris system. Given $\mathcal{A}$ we attempt to partition M into a finite bisimilarity quotient $M/\sim$. If the attempt is successful, then checking reachability of various elements of $\mathcal{A}$ can be directly done on the finite transition system $M/\sim$. Even though an algorithm computing bisimulations may not, in general, terminate, it may be feasible to guarantee termination for certain classes of vector fields and sets. In order to tackle these very interesting questions, we will use the framework of subanalytic sets and stratification theory.

3 Subanalytic Sets and Stratifications

3.1 Real analytic functions, Manifolds, and Stratifications

In this section we describe some fundamental properties of *subanalytic sets*. We concentrate on properties which are useful for the purpose of constructing a bisimulation for the flow of a real analytic vector field. The most important result here is the *Stratification Theorem* (Theorem 2). For this and other important results on subanalytic sets the main references are [5, 12, 16]. We begin by recalling several standard concepts and facts (see [6] and [9] for more details). In this paper, "manifold" means finite-dimensional, Hausdorff, second countable manifold. We say a manifold is *real analytic* (C^ω) if the transition maps between local charts are analytic functions on their domains (which are open subsets of $\mathbb{R}^n$). An *embedded submanifold* S of a manifold M is a topological subspace of M together with a differentiable structure such that the inclusion from S into M is a smooth immersion (i.e. has full rank at every point). A vector field X on the real analytic manifold M is *analytic* if its coordinates in any local chart are analytic. If X is an analytic vector field then any integral curve of X is analytic.

We are interested in intersection properties of sets. From this point of view, infinitely differentiable (C^∞-) functions are not sufficiently "nice". For example, it is not hard to construct a C^∞-function whose zero set is a Cantor-like set. (In fact, *any* closed subset of $\mathbb{R}$ is the zero set of some C^∞-function.) On the other hand, real analytic functions are free from such pathologies. The following classical result illustrates this point.

Theorem 1. *Let I be an open interval and $f: I \to \mathbb{R}$ be an analytic function. Let $Z = \{x \in I | f(x) = 0\}$. Then either $Z = I$ or Z has no accumulation point in I. Equivalently, if f is not identically zero, then every compact subset of I contains at most a finite number of zeros of f.*

Definition 1. *Let M be a real analytic manifold. An* analytic (C^ω) stratification *of M is a partition $\mathcal{S}$ of M with the following properties:*

1. *each $S \in \mathcal{S}$ is a connected, real analytic, embedded submanifold of M,*
2. *$\mathcal{S}$ is locally finite (i.e. every compact subset of M intersects at most finitely many sets in $\mathcal{S}$),*
3. *given two sets $S, P \in \mathcal{S}$, $P \neq S$, such that $S \cap \overline{P} \neq \emptyset$ then $S \subset \overline{P}$ and $\dim S < \dim P$. (We denote by $\overline{P}$ the closure of P.)*

The sets in a stratification are called strata.

3.2 Semianalytic and subanalytic sets

Let M and N be real analytic manifolds and let $C^\omega(M,N)$ denote the set of analytic functions from M into N. (If $f \in C^\omega(M,N)$ we say f is of class C^ω.) Given an analytic manifold U, we denote by $\Sigma(C^\omega(U,\mathbb{R}))$ the Boolean algebra generated by the sets of the form $\{x : f(x) = 0\}$ or $\{x : f(x) > 0\}$, where $f \in C^\omega(U,\mathbb{R})$.

Definition 2. *Let M be a real analytic manifold. A subset A of M is* semianalytic in M *if for every $p \in M$, there is an open neighborhood U of p in M such that $U \cap A \in \Sigma(C^\omega(U,\mathbb{R}))$. If $A \subset M$ is semianalytic in M we write $A \in \mathrm{SMAN}(M)$.*

Definition 3. *Let M be a real analytic manifold. Define $\mathrm{SBAN}_{rc}(M)$ and $\mathrm{SBAN}(M)$ by*

1. *$A \in \mathrm{SBAN}_{rc}(M)$ if and only if there is (N, f, A^*) such that N is a real analytic manifold, $f \in C^\omega(N,M)$, $A^* \in \mathrm{SMAN}(N)$, A^* is relatively compact and $A = f(A^*)$;*
2. *$A \in \mathrm{SBAN}(M)$ if and only if A is a locally finite union of members of $\mathrm{SBAN}_{rc}(M)$.*

We say that A is *subanalytic in M* if $A \in \mathrm{SBAN}(M)$. It is easy to see that $A \in \mathrm{SBAN}_{rc}(M)$ if and only if A is subanalytic in M and relatively compact. The following properties of subanalytic sets are easily derived from the definitions.

1. $\mathrm{SBAN}(M)$ is closed under locally finite unions and intersections.
2. If $A \in \mathrm{SBAN}(M)$ and $f\colon M \longrightarrow N$ is of class C^ω and proper on $\overline{A}$, then $f(A) \in \mathrm{SBAN}(N)$. (A function f is *proper* if $f^{-1}(K)$ is compact whenever K is.)
3. If $A \in \mathrm{SBAN}(N)$ and $f\colon M \longrightarrow N$ is of class C^ω, then $f^{-1}(A) \in \mathrm{SBAN}(M)$.

The following two properties require more subtle proofs. They can be derived from the the stratification theorem for subanalytic sets.

4. If $A \in \mathrm{SBAN}(M)$ then $M \setminus A \in \mathrm{SBAN}(M)$.
5. A subanalytic set has a locally finite number of connected components, each of which is subanalytic.

Example 1. Points are subanalytic, and so is any locally finite union of points, for example $\mathbb{Z}^n$ as subset of $\mathbb{R}^n$. The empty set and M are both in $\mathrm{SBAN}(M)$. Let a, $b \in \mathbb{R}$, $a < b$, then $[a,b]$, $[a,b)$, $(a,b]$ and (a,b) are subanalytic in $\mathbb{R}$. The open ball $B(p,r)$ centered at p of radius r in $\mathbb{R}^n$ is in $\mathrm{SBAN}(\mathbb{R}^n)$.

Example 2. In general, as is clear from the definition, $\mathrm{SMAN}(M)$ is contained in $\mathrm{SBAN}(M)$. In particular, any semialgebraic subset of $\mathbb{R}^n$ is in $\mathrm{SBAN}(\mathbb{R}^n)$.

The following properties clarify further the relation between subanalytic sets and their ambient space. Here we assume that M is a real analytic manifold.

6. Let N be a C^ω, embedded submanifold of M. Then $A \in \mathrm{SBAN}(M) \Longrightarrow A \cap N \in \mathrm{SBAN}(N)$.
7. Let N be as in (5). Let $A \subset N$ be relatively compact and $\overline{A} \subseteq N$. Then $A \in \mathrm{SBAN}(N) \Longrightarrow A \in \mathrm{SBAN}(M)$.
8. For every $p \in M$ and every neighborhood W of p, there exists an open neighborhood V_p of p such that: (a) V_p is relatively compact, (b) $\overline{V}_p \subset W$, and $V_p \in \mathrm{SBAN}(M)$.

Remark 1. Let N be a C^ω, embedded submanifold of M. Then if $A \in \mathrm{SBAN}(N)$ and $N \in \mathrm{SBAN}(M)$ it does not follow that $A \in \mathrm{SBAN}(M)$, as the following example shows.

Example 3. Consider the set $S = \{\frac{1}{n} : n \in \mathbb{N}\}$. As a subset of the open interval $(0, \infty)$ the set S is subanalytic since every compact subset of $(0, \infty)$ intersects S in finitely many points. However, as a subset of $\mathbb{R}$ it is not subanalytic. (See Theorem 1.)

Theorem 2 (Stratification Theorem). *Let M be a real analytic manifold and $\mathcal{A} \subset \mathrm{SBAN}(M)$, $\mathcal{A}$ locally finite. Then there is a C^ω stratification $\mathcal{S}$ of M such that:*

1. *$\mathcal{S} \subset \mathrm{SBAN}(M)$,*
2. *$\mathcal{S}$ is compatible with $\mathcal{A}$. That is, every set in $\mathcal{A}$ is a union of strata from $\mathcal{S}$.*

Remark 2. It is possible to obtain stratifications in which the strata have additional properties. We mention one here which will be useful in the proof of Theorem 5. A *block* in M is a relatively compact, connected, C^ω, embedded submanifold S of M such that there exists a C^ω surjective diffeomorphism $\phi\colon C \to S$ — where C is the open unit cube in $\mathbb{R}^k$, and $k = \dim S$ — such that the graph of ϕ is subanalytic in $\mathbb{R}^k \times M$. We will assume from now on that the strata in $\mathcal{S}$ are blocks.

The following theorem is very useful in proving that certain sets are subanalytic.

Theorem 3. *Consider any formula F of first order predicate calculus with free variables $x_1, \ldots, x_n$ in analytic manifolds $M_1, \ldots, M_n$, which is obtained from formulae in some set $\mathcal{F}$ that involve the x_i and other variables y_j ($y_j \in N_j$, N_j an analytic manifold) by means of the logical operations of conjunction, disjunction, negation, universal and existential quantification. Suppose that the quantifications are* locally bounded *(i.e., that every time a quantifier Qx_i occurs, with $Q = \exists$ or $Q = \forall$, then, if $S_Q(x_i, y)$ is the scope of Qx_i and y are the other variables that are free in S_Q, it follows that for every compact set K of the y domain there is a compact J of the x_i domain such that, for each $\overline{y} \in K$, "$(Qx_i)S_Q(x_i, \overline{y})$" is satisfied if and only if "$(Qx_i \in J)S_Q(x_i, \overline{y})$" is satisfied). Then if the formulae in $\mathcal{F}$ define subanalytic sets, so does F.*

The theorem is simply a consequence of the closure properties of the class of subanalytic sets under Boolean operations and taking direct and inverse images (provided that in the case of direct images the map is proper, see [16]). In view of this result one can, in many cases, prove that a set is subanalytic by writing its definition. The following is an example.

Proposition 1. *Let X be an analytic vector field on $\mathbb{R}^n$. Let $S \subseteq \mathbb{R}^n$ be a C^ω, embedded submanifold, which is also a subanalytic set. Let $\Gamma = \{q \in S \colon X(q) \in T_qS\}$ (here T_qS is the tangent space to S at q). Then Γ is subanalytic in $\mathbb{R}^n$.*

Proof. We can write $\Gamma = X^{-1}(T_qS)$ and a tangent vector (q, v) is in T_qS if and only if

$$\begin{aligned} &q \in S \wedge (v = 0 \vee \\ &\big(\forall \varepsilon \ \ (0 < \varepsilon < 1) \Longrightarrow \big(\exists p \ \ \big(p \in S \wedge p \neq q \wedge \big(\exists r > 0 \ \exists s > 0 \ \ \big(r^2 = \|p - q\|^2 \\ &\qquad\qquad \wedge\, s^2 = \|v\|^2 \wedge \|s(p-q) - rv\|^2 < \varepsilon^2 r^2 s^2 \wedge \|q - p\|^2 < \varepsilon^2\big)\big)\big)\big)\big)\big) \end{aligned}$$

Moreover, as long as q remains in a compact set, the variables p, r, and s can be restricted to lie in a compact set. □

The following proposition can be proved similarly and will be used in the proof of the Theorem 5.

Proposition 2. *Let $\phi \colon (0,1) \to R$ be a C^ω surjective diffeomorphism with $R \subset SBAN_{rc}(\mathbb{R}^n)$. Then $\lim_{s\to 0} \frac{\dot\phi(s)}{\|\dot\phi(s)\|}$ and $\lim_{s\to 1} \frac{\dot\phi(s)}{\|\dot\phi(s)\|}$ exist.*

A deeper and more central result for our analysis is the following. For a proof see [15].

Theorem 4. *Let $\mathcal{A}$ be a locally finite family of nonempty subanalytic subsets of a real analytic manifold M. For each $A \in \mathcal{A}$, let $F(A)$ be a finite set of real analytic vector fields on M. Then there exists a subanalytic stratification $\mathcal{S}$ of M, compatible with $\mathcal{A}$, and having the property that, whenever $S \in \mathcal{S}$, $S \subset A$, $A \in \mathcal{A}$, $X \in F(A)$, then either (i) X is everywhere tangent to S or (ii) X is nowhere tangent to S.*

We finish this section with a simple proposition which illustrates some of the good intersection properties that analytic curves have with subanalytic sets. The "finiteness" property indicated in the proposition makes it possible to define transitions between strata in a natural way.

Proposition 3. *Let I be an open interval, M a real analytic manifold and $\gamma \colon I \to M$ a real analytic function. Let $\mathcal{S}$ be a C^ω stratification of M by subanalytic sets (that is, $S \in \mathcal{S} \Rightarrow S \in \mathrm{SBAN}(M)$). If $[a, b] \subset I$ then there exists a finite partition $\{x_1, \ldots, x_n\}$ of $[a, b]$ with the property that for each $i = 1, \ldots, n-1$ there exists a stratum $S_i \in \mathcal{S}$ such that $\gamma((x_i, x_{i+1})) \subseteq S_i$.*

Proof. Consider the family $\mathcal{I} = \{\gamma^{-1}(S) \cap [a,b] \colon S \in \mathcal{S}\}$. Since $\gamma([a,b])$ is compact and $\mathcal{S}$ is locally finite, the family $\mathcal{I}$ is a finite partition of $[a,b]$. By Property 3 of subanalytic sets the sets in $\mathcal{I}$ are subanalytic in I. By Theorem 2, there exists a C^ω stratification $\mathcal{J}$ of $[a,b]$ compatible with $\mathcal{I}$. Therefore, $\mathcal{J}$ consists of a finite number of points and open intervals. Moreover, for each $J \in \mathcal{J}$ there exists $S \in \mathcal{S}$ such that $\gamma(J) \subseteq S$, as desired. □

Example 4. The assumption of subanalyticity in the proposition can not be dropped. Consider the stratification of $\mathbb{R}^2$ by the following five sets:

$$
\begin{aligned}
S_1 &= \{(0,0)\} \\
S_2 &= \left\{(x,y) \colon x > 0 \ \wedge \ y = x \sin \frac{1}{x}\right\} \\
S_3 &= \left\{(x,y) \colon x < 0 \ \wedge \ y = x \sin \frac{1}{x}\right\} \\
S_4 &= \left\{(x,y) \colon x \neq 0 \ \wedge \ y > x \sin \frac{1}{x}\right\} \bigcup \{(0,y) \colon y > 0\} \\
S_5 &= \left\{(x,y) \colon x \neq 0 \ \wedge \ y < x \sin \frac{1}{x}\right\} \bigcup \{(0,y) \colon y < 0\}
\end{aligned}
$$

Notice that S_1, S_2 and S_3 form the graph of the function $f(x) = x \sin \frac{1}{x}$ ($f(0) = 0$), while S_4 and S_5 denote the region above and the below the graph, respectively. Each set is a C^ω, embedded submanifold of $\mathbb{R}^2$ and they clearly satisfy the condition on the dimension of the strata in the closure of other strata. Finally, consider the constant vector field $X = \frac{\partial}{\partial x}$. Then the integral curve γ of X through $(0,0)$ is the x-axis (parameterized by x itself). Therefore, the image by γ of any interval containing 0 intersects both S_4 and S_5 an infinite number of times.

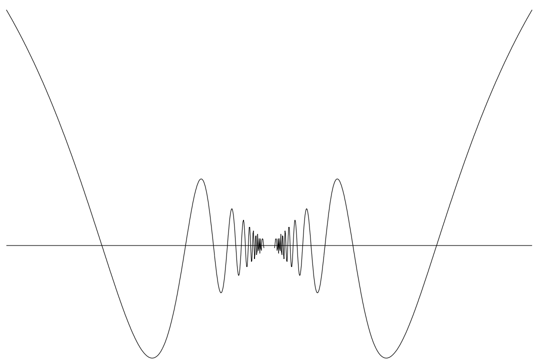

Fig. 1. Infinite crossings on a compact interval

4 Bisimulations of Analytic Vector Fields

Here we describe a process for the construction of a bisimulation for the flow of a real analytic vector field. We assume that we are given a real analytic vector field X on a connected real analytic manifold M as well as a finite family $\mathcal{A}$ of relatively compact subanalytic sets. These sets may describe initial conditions, guards, invariants or undesirable regions of the continuous evolution within a discrete location of a hybrid automaton.

We now invoke Theorem 4 (here there is a single vector field on every stratum) to obtain a stratification $\mathcal{S}$ of M by subanalytic sets which is compatible with $\mathcal{A}$ and such that on each stratum X is either everywhere tangent or nowhere tangent. More precisely, for each $S \in \mathcal{S}$ either: (1) for all q in S, X is tangent to S at q, or (2) for all q in S, X is not tangent to S at q. We now wish to study how the integral curves of X *enter* and *leave* each stratum of $\mathcal{S}$. For this we need a more precise definition of what we mean by *entering* and *leaving* a stratum.

Definition 4. *Given two subsets S, T of M, and a real analytic curve $\gamma : I \to M$ (I an open interval), we say that γ* leaves S through T *(or* enters T from S*) if one of the following exiting conditions is satisfied:*

E1 *there exist $a, b \in I$ such that $\gamma(t) \in S$ for all $t \in (a, b)$ and $\gamma(b) \in T$*
E2 *there exist $a, b \in I$ such that $\gamma(a) \in S$ and $\gamma(t) \in T$ for all $t \in (a, b)$.*

The following proposition shows that this definition covers all possible "exiting" situations for strata of $\mathcal{S}$.

Proposition 4. *Let $S \in \mathcal{S}$ and γ be as above. If there exists $t_0, t_1 \in I$ such that $\gamma(t_0) \in S$ and $\gamma(t_1) \notin S$ then there is a stratum T such that either* **E1** *or* **E2** *holds.*

It is clear from Definition 4 that in case **E1**, $T \cap \overline{S} \neq \emptyset$. By property 3 of a stratification, we conclude $T \subset \overline{S}$ and $\dim T < \dim S$. Similarly in case **E2**, $S \subset \overline{T}$ and $\dim S < \dim T$.

Definition 5. *We call a stratum $S \in \mathcal{S}$* tangential *if the vector field X is tangent to S at every point of S. We call a stratum* transversal *otherwise.*

The following proposition clarifies further the possible exit situations.

Proposition 5. *Let S, T be strata in $\mathcal{S}$ and γ an integral curve of X which leaves S through T. Then one (and only one) of the following holds:*

1. *condition* **E1** *holds, S is a tangential stratum and T is a transversal stratum.*
2. *condition* **E2** *holds, S is a transversal stratum and T is s tangential stratum.*

Our goal is to construct a bisimulation as a quotient of the equivalence relation induced by the stratification $\mathcal{S}$. More precisely, we would like to define the equivalence relation $\sim_{\mathcal{S}}$ by $p \sim_{\mathcal{S}} q$ iff p,q belong to the same stratum of $\mathcal{S}$. In $M/\sim_{\mathcal{S}}$ there is a transition from the stratum S to the stratum T iff an integral curve of X leaves S through T. In order to obtain a bisimulation we need the stratification $\mathcal{S}$ to satisfy the following two conditions:

1. if an integral curve of X starting at a point of the stratum S does not exit S, then no other integral curve starting in S leaves S,
2. whenever an integral curve of X which starts in S leaves the stratum through T, then all other integral curves which start in S leave the stratum through T.

In order to satisfy those conditions we refine the stratification further according to exit features of the integral curves. We describe the iterative process below, which is analogous to the bisimulation algorithm described in Section 2. If the process terminates we obtain the desired bisimulation.

Definition 6 (Refinement Process). *The process has two steps which will need to be iterated. In the first step we refine the tangential strata and in the second we refine the transversal strata.*

Step 1 *Let S be a tangential stratum. For each $T \in \mathcal{S}$, $T \subset \overline{S}$, $T \neq S$ let S_T denote the set of points $q \in S$ for which the integral curve of X through q leaves S through T. Let $S_0 = S \setminus \cup S_T$, where the union is taken over all strata T contained in $\overline{S}$ and different from S. So, S_0 is the set of points $q \in S$ such that the integral curve of X through q at time $t = 0$, remains in S for all $t \geq 0$. We subdivide S into the sets S_T and S_0. This is a finite subdivision of S.*

Step 2 *Let R be a transversal stratum. Let $R^b = \{S \in \mathcal{S} : S \neq R, R \subset \overline{S}\}$. For each $S \in R^b$ and $T \subset \overline{S}$ ($T \neq S$), let R_{S_T} be the set of points $r \in R$ such that the integral curve through r leaves R through S_T. Also, let R_{S_0} denote the set of points $r \in R$ such that the integral curve through r leaves R through S_0. We subdivide R into the sets R_{S_T}, R_{S_0} where S varies over R^b. This is a finite subdivision of R.*

Remark 3. The new subdivision sets from Step 1 and Step 2 are not in general subanalytic. Therefore, Step 2 requires some clarification since we claim that trajectories "leave R through" one of the sets S_T or S_0. According to definition 4 we need to verify either **E1** or **E2**. The following proposition gives the key argument.

Proposition 6. *Assume the tangential stratum S is subdivided as in Step 1. Let $\gamma : [t_0, t_1] \to M$ be an integral curve of X such that $\gamma(t) \in S$ for $t_0 \leq t \leq t_1$. If there exists a set S_* resulting from the subdivision of S such that $\gamma(t_0) \in S_*$ and $\gamma(t_1) \in S_*$ then $\gamma(t) \in S_*$ for $t_0 \leq t \leq t_1$.*

Proof. It follows immediately from the definition of the sets in Step 1, since once a point of a trajectory is in one such set, then for as long as the trajectory remains in S it will belong to the same set. □

Notation: we will write γ_q to denote the integral curve of X which passes through q at time 0, i.e. with $\gamma_q(0) = q$.

Proposition 7. *With S and R as above, for each $q \in R$ such that γ_q leaves R through S, there exists S_* ($S_* = S_T$ for some T or $S_* = S_0$) such that γ_q leaves R through S_*.*

Proof. Let $\gamma_q : I \to M$, $a, b \in I$ be such that $\gamma(a) \in R$ and $\gamma(t) \in S$ for $a < t < b$ (we are assuming R is transversal so **E2** holds). Let $S_1, \ldots, S_k$ be the sets in the subdivision of S given by Step 1. Let $t_i = \inf\{t \in (a,b) : \gamma_q(t) \in S_i\}$. Then $a = \min\{t_i\} = t_{i_0}$ for some i_0. We claim that γ_q leaves R through S_{i_0}. To see this let $s_1 \in (a,b)$ be such that $\gamma_q(s_1) \in S_{i_0}$. Suppose there is s with $a < s < s_1$ and $\gamma_q(s) \in S_j$ for $j \neq i_0$. Then there exists s_0 with $a < s_0 < s < s_1$ and $\gamma_q(s_0) \in S_{i_0}$. But this contradicts the previous proposition, so we must have $\gamma_q(s) \in S_{i_0}$ for $a < s \leq s_1$. □

Notice that in Step 2 we may be subdividing some sets which are in the closure of some tangential set. This requires the iteration of the two steps. In general, we should not expect this process to terminate even if we deal with a finite number of strata or if we limit our study to a compact set. The following example illustrates this point (see Figure 2).

Example 5. Let $M = \mathbb{R}^2$ and X be the linear vector field $\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \mathbf{x}$. Assume the stratification consists of the following five strata: $S_1 = \{(0,0)\}$, $S_2 = \{(4,0)\}$, $S_3 = \{(x,0) : 0 < x < 4\}$, $S_4 = \{(x,0) : x > 4\}$, and $S_5 = \mathbb{R}^2 \setminus \cup_{i=1}^{4} S_i$. The

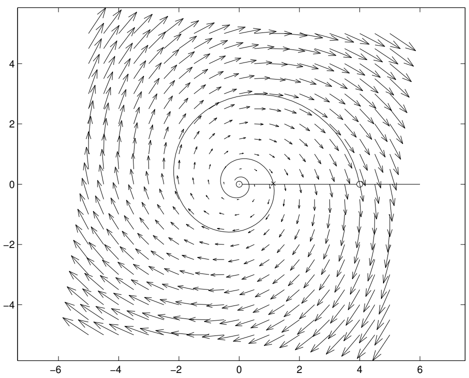

Fig. 2. Process does not terminate

integral curves of X are spirals moving away from the origin. Here S_1 and S_5 are tangential strata. The others are transversal strata. There is no subdivision possible (or necessary) for S_1. The curves through S_5 exit at one of the three strata S_2, S_3, or S_4. Step 1 requires that we subdivide S_5 into three regions. Two regions are composed of (parts of) the integral curves of X which exit S_5 through S_3 and S_4 respectively. The third is composed of (a part of) the single integral curve which exits through the point S_2. Step 2 now requires that we subdivide the transversal strata according to a similar rule, but now curves from S_3 leave through three different regions and we must subdivide this stratum further (into three regions, in fact). The subdivision point corresponds to the first point of intersection of S_3 and the integral curve from S_2 run backwards in time. This now causes one of the regions in S_5 to be subdivided further and clearly the process will not terminate.

For linear vector fields on the plane, the existence of "spiral" points, such as above, is the only obstruction to the procedure as the following theorem illustrates.

Theorem 5. *Let $M = \mathbb{R}^2$, X be the linear vector field Ax and assume that the eigenvalues of A are either real or purely imaginary. Let K be a compact set and define $\mathcal{S}_K = \{S \in \mathcal{S} : S \cap K \neq \emptyset\}$ (which is therefore finite). Then the Refinement Process applied to $\mathcal{S}_K$ terminates.*

Proof. We will carry out the proof in more generality than necessary. See the remark below.

If $A = 0$ then the process terminates with Step 1, since for each 2-dimensional S, we have $S = S_0$. If $A \neq 0$, then the zero set of A is either $\{(0,0)\}$ or a line through the origin. We will deal in detail with the first case. The second case can be analyzed with similar methods. We will assume that $\{(0,0)\}$ is a stratum of $\mathcal{S}$ ($\mathcal{S}$ can be made compatible with $\{(0,0)\}$). This implies that tangential 1-dimensional strata will not be subdivided further since such a stratum is an arc of a single trajectory of X. Hence, we only need to study the tangential 2-dimensional strata and the transversal 1-dimensional strata (there are no transversal 2-dimensional strata).

The first iteration of the process requires a special analysis. Let $R, S \in \mathcal{S}$, S 2-dimensional and R 1-dimensional, transversal and contained in $\overline{S}$. Let S_0 be as in Step 1 and R_{S_0} as in Step 2. The following lemma characterizes R_{S_0}.

Lemma 1. *There is a finite stratification of R_{S_0} by subanalytic subsets of $\mathbb{R}^2$.*

Proof. Since R is a block (see Remark 2), there is a C^ω diffeomorphism $\phi : (0,1) \to R$ whose graph is subanalytic in $\mathbb{R} \times \mathbb{R}^2$. On the interval $(0,1)$ the connected sets are intervals and hence subanalytic in $\mathbb{R}$. Therefore, connected sets in R are subanalytic and their boundary (in R) consists of at most two points. To prove the lemma we show that R_{S_0} has finitely many connected components.

Suppose to the contrary that R_{S_0} has infinitely many connected components. Then there exists two infinite sequences $\{s_i\}$, $\{t_i\}$ in the interval $(0,1)$ such that, for all i, $s_i < t_i < s_{i+1}$, $\phi(s_i) \in R_{S_0}$, and $\phi(t_i) \notin R_{S_0}$. For each i, consider the curve C_i made up of the arcs $\phi([s_i, s_{i+1}])$, $\Gamma_i = \{\gamma_{\phi(s_i)}(t) : t \geq 0\}$, $\Gamma_{i+1} = \{\gamma_{\phi(s_{i+1})}(t) : t \geq 0\}$, and the point $(0,0)$. This is a continuous, closed, simple curve and therefore it divides the plane into two open connected sets. The trajectory $\gamma_{\phi(t_i)}$ enters one of these sets. This trajectory does not intersect the others and it can not pass through $(0,0)$. Moreover, it can not leave the region through R since all trajectories cross R in the same direction. Therefore, $\gamma_{\phi(t_i)}(t)$ must remain in the same region for all $t > 0$. On the other hand, by construction $\gamma_{\phi(t_i)}$ must leave S. Let $\bar{t}$ be the first t such that $\gamma_{\phi(t_i)}(t) \notin S$. So, $q_i = \gamma_{\phi(t_i)}(\bar{t}) \in \overline{S} \setminus S$. The set $S \cup R \cup \{(0,0)\}$ is connected, relatively compact, subanalytic, and contains each C_i. By construction each q_i is in a different connected component of $\overline{S} \setminus S$, but this is a contradiction since $\overline{S} \setminus S$ has a finite number of connected components. Therefore, so does R_{S_0} as desired. □

Since the sets S_0 will not be subdivided in subsequent iterations of the refinement process, neither will the sets R_{S_0}. If the stratum $T \in \mathcal{S}$, $T \subset \bar{S}$ is 0-dimensional (i.e. T is a singleton) then R_{S_T} is also 0-dimensional (by uniqueness of solutions). The following lemma characterizes the sets R_{S_T} when T is a 1-dimensional stratum.

Lemma 2. *If $T \in \mathcal{S}$ is 1-dimensional then R_{S_T} is open in R.*

Proof. Let $q \in R_{S_T}$. The trajectory γ_q enters S_T and leaves S through T. Let t_1 be the smallest value of t such that $\gamma_q(t) \in T$. Set $p = \gamma_q(t_1)$. Since X is transversal to R and T we can find relatively open (connected) neighborhoods N_R of q in R, N_T of p in T, and $\delta > 0$, such that $V = \{\gamma_y(t) : y \in N_R, |t| < \delta\}$ and $W = \{\gamma_y(t) : y \in N_T, |t| < \delta\}$ are open, $V \cap W = \emptyset$, $V \cap R = N_R$, and $W \cap T = N_T$. Moreover, we can choose the above so that $\{\gamma_y(t) : y \in N_R, 0 < t < \delta\} \subset S \cap V$ and $\{\gamma_y(t) : y \in N_T, -\delta < t < 0\} \subset S \cap W$. (These constructions are a consequence of basic theorems on differential equations.) We can also assume that the only strata of dimension 0 or 1 which intersect $S \cup V \cup W$ are R and T (because $\mathcal{S}$ is a stratification). Notice that if γ is a trajectory of X such that $\gamma(s) \in W$, then there exists s' such that $\gamma(s') \in W \cap T$ and $\gamma(t) \in W$ for all t between s and s'. By continuous dependence on initial conditions, for all $\varepsilon > 0$ there exist a neighborhood $\tilde{N}_R \subset N_R$ of q in R and $0 < \tilde{\delta} < \delta$ such that $|\gamma_q(t) - \gamma_y(t)| < \varepsilon$, for all $y \in \tilde{V} = \tilde{N}_R \times (-\tilde{\delta}, \tilde{\delta})$ and $0 \leq t \leq t_1$. The set $\Gamma = \{\gamma_q(t) : 0 \leq t \leq t_1\}$ is compact and contained in the open set $S_e = S \cup \tilde{V} \cup W$. Set $d = \frac{1}{3}\mathrm{dist}(q, \{\gamma_q(t) : \delta \leq t \leq t_1\})$. We choose $\varepsilon > 0$ so that $\varepsilon < d$, $\{r : \mathrm{dist}(r, \Gamma) < \varepsilon\} \subset S_e$, and the ball, $B(p, \varepsilon)$, of center p and radius ε, is contained in W. Then for all $y \in \tilde{N}_R$, $\gamma_y(t) \in S_e$ for $0 \leq t \leq t_1$ and $\gamma_y(t_1) \in B(p, \varepsilon)$. We assume further, that $\tilde{N}_R \subset B(q, \varepsilon)$. Let $t_2 = \inf\{t > 0 : \gamma_y(t) \in W\}$ (so $t_2 > 0$ because $V \cap W = \emptyset$). We claim that if $y \in \tilde{N}_R$, then $\gamma_y(t) \in S$ for all $0 \leq t \leq t_2$. Suppose to the contrary that there exists t with $0 < t \leq t_2$ and $\gamma_y(t) \notin S \cup W$. Let $\bar{t} = \inf\{t : 0 < t \leq t_2, \gamma_y(t) \notin S \cup W\}$. Since $\gamma_y(t) \in S$ for $0 < t < \delta$, we know that $\bar{t} \geq \delta > 0$. On the other hand, $\gamma_y(\bar{t}) \in \tilde{V}$. Therefore, $\gamma_y(\bar{t}) \in \overline{S} \cap S_e$ and so $\gamma_y(\bar{t}) \in R \cap \tilde{V} = \tilde{N}_R \subset B(q, \varepsilon)$. But then $\mathrm{dist}(\gamma_q(\bar{t}), q) \leq \mathrm{dist}(\gamma_q(\bar{t}), \gamma_y(\bar{t})) + \mathrm{dist}(\gamma_y(\bar{t}), q) < 2\varepsilon$, which contradicts the choice of ε. Hence, we have $\gamma_y(t) \in S$ for $y \in \tilde{N}_R$ and $0 < t \leq t_2$. By the definition of t_2 and because S is open we can find $\bar{\varepsilon} > 0$ such that $\gamma_y(t) \in S$ for $0 < t < t_2 + \bar{\varepsilon}$ and $\gamma_y(t_2 + \bar{\varepsilon}) \in W$. This implies that γ_y must exit S through T, i.e. $y \in R_{S_T}$. Therefore, $\tilde{N}_R \subset R_{S_T}$ and R_{S_T} is open in R. □

We continue with the main proof. For a 1-dimensional stratum T we can write $T = \cup_{i=1}^{\infty} \phi(J_i)$, with J_i open intervals. For each endpoint a_i of J_i, $x_i = \phi(a_i)$

is in the complement of R_{S_T}. So, either $\{x_i\} = R_{S_{T_i}}$ for some 0-dimensional stratum T_i, or x_i is in the relative boundary of R_{S_0} in R. In either case, there are only finitely many such points x_i. It follows that R_{S_T} has a finite number of connected components.

We have shown that after one iteration we obtain a finite number of singleton sets (points) $\mathcal{P}^{(1)}$ and 1-dimensional connected subanalytic sets $\mathcal{A}^{(1)}$ such that all the new subdivision sets are unions of sets in $\mathcal{P}^{(1)} \cup \mathcal{A}^{(1)}$. Subsequent iterations will only depend on sets of the form R_{S_T} with $T \in \mathcal{P}^{(1)} \cup \mathcal{A}^{(1)}$ (not on R_{S_0}). In fact, if we define for $n \geq 2$, $\mathcal{P}^{(n)}$ as the collection of sets R_{S_T} with $T \in \mathcal{P}^{(n-1)}$, then subsequent subdivisions depend on $\mathcal{P}^{(n)} \neq \emptyset$ (that is, if $\mathcal{P}^{(n)} = \emptyset$ then the Refinement Process terminates). Such sets R_{S_T} are always singletons and there is always a finite number of them. Moreover, by its definition, for each point p with $\{p\} \in \mathcal{P}^{(n)}$ there is a unique $\{q\} \in \mathcal{P}^{(1)}$ and $t_p < 0$ such that $p = \gamma_q(t)$.

Suppose now that the Refinement Process does not terminate. Then there is a trajectory γ of X and an infinite sequence of points x_n with $\{x_n\} \in \mathcal{P}^{(n)}$, and $x_n = \gamma(t_n)$ with $t_n \to -\infty$. Since we are only subdividing sets in $\mathcal{S}_K$ the sequence must stay in a compact set. We may assume that $\{x_n\}$ converges to a point x_0. Consider first the case when the eigenvalues of A are real. Then $x_0 = (0,0)$. Moreover, there exists a one dimensional stratum R of $\mathcal{S}$ which contains infinitely many x_n's and therefore $(0,0) \in \overline{R}$. Assume that R is diffeomorphic to the interval $(0,1)$ via ϕ as above, with $\lim_{s\to 0} \phi(s) = (0,0)$. By Proposition 2 there exists $v = \lim_{s\to 0} \frac{\dot{\phi}(s)}{\|\dot{\phi}(s)\|}$. A direct calculation shows that the following limit also exists: $w = \lim_{t_n\to -\infty} \frac{\dot{\gamma}(t_n)}{\|\dot{\gamma}(t_n)\|}$. We must have $v \neq -w$, otherwise R can not intersect γ for $|t_n|$ large enough. By changing coordinates and restricting the study to a neighborhood of $(0,0)$ we may assume that $\{\gamma(t) : t < t_0\}$ and R are both graphs of functions, ψ_γ and ψ_R respectively, with domain $(0,1)$. At two consecutive intersections s_1, s_2 of these graphs, the vector X must point to opposite sides of the graph of ψ_R. By continuity, for some s, $s_1 < s < s_2$ the vector $X(\psi_R(s))$ must be tangent to the R. This contradicts the transversality of R. Therefore, R and γ can not intersect near $(0,0)$ and the Refinement Process must terminate. In case A has purely imaginary eigenvalues, $x_0 \neq (0,0)$ since all trajectories are periodic. Still a similar argument applies using v as above and $w = X(x_0)$. This concludes the proof of the main theorem. $\square$

Remark 4. The same proof extends to analytic vector fields on the plane with isolated equilibria and with the property that bounded trajectories are either periodic or have "limit" directions (the vector w in the proof). The existence of those limit directions was the only part in the proof that used the linearity of X in an essential way.

5 Conclusions

We presented some preliminary results on obtaining finite bisimulations of analytic vector fields. An algorithm is provided and termination is guaranteed for a class of linear vector fields.

Even though in this paper continuous dynamic systems were considered, the extensions to hybrid systems, even though harder, are conceptually similar. Bisimulations of hybrid systems can still be considered in the framework of subanalytic stratifications by allowing multiple vector fields as well as reset maps. However, the reset maps must be in some sense compatible with the flows for the procedure to terminate. This requirement is already necessary when dealing with a timed automata where the clocks run with irrational slopes.

It should be noted that the main results of this paper are existential since they prove the existence of finite bisimulations. However, there is a long way to making this procedure computationally effective. For certain classes of vector fields the construction can be made effective. For example, one could generalize the decidability result in [8] for multi-polynomial vector fields on the plane, by effectively constructing a finite bisimulation using techniques similar to the ones in this paper (allowing for semialgebraic sets instead of just polyhedra).

Furthermore, if the bisimulation algorithm does not terminate (or is not computable), it may be useful to consider system overapproximations, or abstractions [14], for which the algorithm would terminate (or can be computed).

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