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Approximate reduction of dynamic systems[☆]

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Abstract

The reduction of dynamic systems has a rich history, with many important applications related to stability, control and verification. Reduction of nonlinear systems is typically performed in an "exact" manner – as is the case with mechanical systems with symmetry – which, unfortunately, limits the type of systems to which it can be applied. The goal of this paper is to consider a more general form of reduction, termed *approximate reduction*, in order to extend the class of systems that can be reduced. Using notions related to incremental stability, we give conditions on when a dynamic system can be projected to a lower dimensional space while providing hard bounds on the induced errors, i.e. when it is behaviourally similar to a dynamic system on a lower dimensional space. These concepts are illustrated on a series of examples.

Keywords: Reduction; Approximate reduction; Dynamical systems; Incremental input-to-state stability

1. Introduction

Modelling is an essential part of many engineering disciplines and often a key ingredient for successful designs. Although it is widely recognized that models are only approximate descriptions of reality, their value lies precisely on the ability to describe, within certain bounds, the modelled phenomena. In this paper we consider modeling of closed-loop nonlinear control systems, i.e. differential equations, with the purpose of simplifying the analysis of these systems. The goal of this paper is to reduce the dimensionality of the differential equations being analysed while providing hard bounds on the introduced errors. One promising application of these techniques is to the verification of hybrid systems, which is currently constrained by the complexity of high dimensional differential equations.

Reducing differential equations – and in particular mechanical systems – is a subject with a long and rich history. The first form of reduction was discovered by Routh in the 1860's; over the years, geometrical reduction has become an academic field in itself. One begins with a differential equation with certain symmetries, i.e. a differential equation invariant under the action of a Lie group on the phase space. Using these symmetries, one can reduce the dimensionality of the phase space (by "dividing" out by the symmetry group) and define a corresponding differential equation on this reduced phase space. The main result of geometrical reduction is that one can understand the behaviour of the full-order system in terms of the behavior of the reduced system and vice versa [11,16,5]. While this form of "exact" reduction is very elegant, the class of systems for which this procedure can be applied is actually quite small. This indicates the need for a form of reduction that is applicable to a wider class of systems and, while not being exact, is "close enough".

In systems theory, reduced order modelling has also been extensively studied under the name of model reduction [4,3]. The typical problem addressed in this literature consists in approximating a system Σ_1 by a system Σ_2 while minimizing the L_2 norm:

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$$\left(\int_0^\infty |y_1(t) - y_2(t)|^2 \mathrm{d}t\right)^{\frac{1}{2}} \tag{1}$$

where y_1 is the output of Σ_1 and y_2 is the output of Σ_2. This kind of reduction is not adequate when one is interested in applications to formal verification of hybrid systems. A typical safety verification problem consists in determining if any trajectory of Σ_1 starting in a given set of initial conditions S enters a given set of unsafe states B. If one solves this verification problem with the reduced order model Σ_2, then one cannot conclude, based on an upper bound on (1), if trajectories of Σ_1 do enter the B. This motivates us to study reduction problems in which trajectories of Σ_1 and its reduced model Σ_2 are instead related by the L_∞ norm:

$$\sup_{t \in [0,\infty[} |y_1(t) - y_2(t)|. \tag{2}$$

More recent work considered exact reduction of control systems [17,15] based on the notion of bisimulation which was later generalized to approximate bisimulation [7,13,8].

We develop our results in the framework of incremental stability and our main result is in the spirit of existing stability results for cascade systems that proliferate the Input-to-State Stability (ISS) literature. See, for example, [12] and the references therein. A preliminary version of our results appeared in the conference paper [14].

2. Preliminaries

A continuous function $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, is said to belong to class $\mathcal{K}_\infty$ if it is strictly increasing, $\gamma(0) = 0$ and $\gamma(r) \to \infty$ as $r \to \infty$. A continuous function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the map $\beta(r, s)$ belongs to class $\mathcal{K}_\infty$ with respect to r and, for each fixed r, the map $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$.

A function $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ is said to be smooth if it is infinitely differentiable. We denote by $T\varphi$ the tangent map to φ and by $T_x\varphi$ the tangent map to φ at $x \in \mathbb{R}^n$. The map φ is said to be a submersion at $x \in \mathbb{R}^n$ if $T_x\varphi$ is surjective and is said to be a submersion if it is a submersion at every $x \in \mathbb{R}^n$. When φ is a submersion we will also use the notation $\ker(T\varphi)$ to denote the distribution:

$$\ker(T\varphi) = \{X : \mathbb{R}^n \to \mathbb{R}^n \mid T\varphi \cdot X = 0\}.$$

The Lie bracket of vector fields X and Y is denoted by $[X, Y]$ and $[\ker T\varphi, Y]$ denotes the distribution defined by all the vector fields Z such that $Z = [X, Y]$ for some $X \in \ker T\varphi$.

Given a point $x \in \mathbb{R}^n$, $|x|$ denotes the usual Euclidean norm while $||f||$ denotes $\operatorname{ess\,sup}_{t\in[0,\tau]} |f(t)|$ for any given function $f : [0, \tau] \to \mathbb{R}^n$, $\tau \in \mathbb{R}^+$.

2.1. Dynamic and control systems

In this paper we shall restrict our attention to dynamic and control systems defined on Euclidean spaces.

Definition 1. A vector field is a pair $(\mathbb{R}^n, X)$ where X is a smooth map $X : \mathbb{R}^n \to \mathbb{R}^n$. A smooth curve $\mathbf{x} : I \to \mathbb{R}^n$, defined on an open subset I of $\mathbb{R}$ including the origin, is said to be a trajectory of $(\mathbb{R}^n, X)$ if the following condition holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = X(\mathbf{x}(t)) \quad \forall t \in I.$$

When we want to emphasize the initial condition $\mathbf{x}(0) = x$ we shall denote a trajectory as $\mathbf{x}(\cdot, x)$. A vector field is said to be forward complete when for every $x \in \mathbb{R}^n$ the trajectory $\mathbf{x}(\cdot, x)$ is defined on an interval of the form $]-a, +\infty[$ for some $a < 0$. *All the vector fields in this paper are assumed to be forward complete*. This assumption is always satisfied for problems of formal verification in which the vector field describes the result of applying a stabilizing controller to the open-loop dynamics.

Definition 2. A control system is a triple $(\mathbb{R}^n, \mathbb{R}^m, F)$ where F is a smooth map $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. A smooth curve $\mathbf{x} : I \to \mathbb{R}^n$, defined on an open subset I of $\mathbb{R}$ including the origin, is said to be a trajectory of $(\mathbb{R}^n, \mathbb{R}^m, F)$ if there exists a smooth input curve $\mathbf{u} : I \to \mathbb{R}^m$ such that the following condition holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = F(\mathbf{x}(t), \mathbf{u}(t)) \quad \forall t \in I.$$

Similarly to vector fields, we denote by $\mathbf{x}_\mathbf{u}(\cdot, x)$ the trajectory $\mathbf{x}$ of a control system associated with the input curve $\mathbf{u}$ and satisfying $\mathbf{x}(0) = x$.

We have defined trajectories based on smooth input curves mainly for simplicity since the presented results hold under weaker regularity assumptions.

3. Exact reduction

For some dynamic systems described by a vector field X on $\mathbb{R}^n$ it is possible to replace X by a vector field Y describing the dynamics of the system on a lower dimensional space, $\mathbb{R}^m$, while retaining much of the information about X. When this is the case we say that X can be reduced to Y. This idea of (exact) reduction is captured by the notion of φ-related vector fields.

Definition 3. Let $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map. The vector field $(\mathbb{R}^n, X)$ is said to be φ-related to the vector field $(\mathbb{R}^m, Y)$ if for every $x \in \mathbb{R}^n$ we have:

$$T_x\varphi \cdot X(x) = Y \circ \varphi(x). \tag{3}$$

The following proposition, proved in [1], characterizes φ-related vector fields in terms of their trajectories.

Proposition 1. *The vector field $(\mathbb{R}^n, X)$ is φ-related to the vector field $(\mathbb{R}^m, Y)$ for some smooth map $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ iff for every $x \in \mathbb{R}^n$ and for every $t \in \mathbb{R}_0^+$ we have:*

$$\varphi \circ \mathbf{x}(t, x) = \mathbf{y}(t, \varphi(x)), \tag{4}$$

where $\mathbf{x}$ and $\mathbf{y}$ are the trajectories of X and Y, respectively.

For φ-related vector fields, we can replace the study of trajectories $\mathbf{x}(\cdot, x)$ with the study of trajectories $\mathbf{y}(\cdot, \varphi(x))$ living on the lower dimensional space $\mathbb{R}^m$. To illustrate the usefulness of this result in the context of formal verification, assume that one is interested in showing that part of the state will never enter a set of undesirable states $B \subseteq \mathbb{R}^m$. If the part of the state we are interested in is given by $y = \varphi(x)$ with $y \in \mathbb{R}^m$, $m < n$ and if X is φ-related to Y then we can analyse the evolution of y by working with the reduced model Y instead of working with the full-order model X.

If a vector field and a submersion φ are given we can use the following result, proved in [10], to determine the existence of φ-related vector fields.

Proposition 2. *Let $(\mathbb{R}^n, X)$ be a vector field and let $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth submersion. There exists a vector field $(\mathbb{R}^m, Y)$ that is φ-related to $(\mathbb{R}^n, X)$ iff:*

$$[\ker(T\varphi), X] \subseteq \ker(T\varphi). \tag{5}$$

When X is a linear vector field $X(x) = Ax$ and φ is a linear map $\varphi(x) = Lx$, condition (5) admits a simpler and intuitive description. Recalling that $[v, Ax] = Av$ for any $v \in \mathbb{R}^n$, (5) becomes $Ak \in \ker(L)$ for every $k \in \ker(L)$ or equivalently, $A(\ker(L)) \subseteq \ker(L)$. A linear vector field Ax would then be L-related to another vector field iff $\ker(L)$ is an A-invariant subspace of $\mathbb{R}^n$. This is easily seen to be a quite restrictive condition. In order to enlarge the class of vector fields that can be reduced we introduce, in the next section, an approximate notion of reduction.

4. Approximate reduction

The generalization of Definition 3 proposed in this section requires a decomposition of $\mathbb{R}^n$ of the form $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^k$. Associated with this decomposition are the canonical projections $\pi_m : \mathbb{R}^n \to \mathbb{R}^m$ and $\pi_k : \mathbb{R}^n \to \mathbb{R}^k$ taking $\mathbb{R}^n \ni x = (y, z) \in \mathbb{R}^m \times \mathbb{R}^k$ to $\pi_m(x) = y$ and $\pi_k(x) = z$, respectively. Intuitively, $\mathbb{R}^n$ corresponds to the state space of the full model and we will be interested in the evolution of only the part of the state described by $y = \pi_m(x)$, for which we will be seeking a reduced model.

Definition 4. The vector field $(\mathbb{R}^n, X)$ is said to be approximately π_m-related to the vector field $(\mathbb{R}^m, Y)$ if there exist a class $\mathcal{K}_\infty$ function γ and a constant $c \in \mathbb{R}_0^+$ such that the following estimate holds for every $x \in \mathbb{R}^n$ and for every $t \in \mathbb{R}_0^+$:

$$|\pi_m \circ \mathbf{x}(t, x) - \mathbf{y}(t, \pi_m(x))| \leq \gamma(|\pi_k(x)|) + c, \tag{6}$$

where $\mathbf{x}$ and $\mathbf{y}$ are the trajectories of X and Y, respectively.

Note that when X and Y are π_m-related we have:

$$|\pi_m \circ \mathbf{x}(t, x) - \mathbf{y}(t, \pi_m(x))| = 0,$$

which implies (6). Definition 4 can thus be seen as a generalization of exact reduction captured by Definition 3. Similar ideas have been used in the context of approximate notions of equivalence for control systems [8].

Although the bound on the gap between the projection of the original trajectory $\mathbf{x}$ and the trajectory $\mathbf{y}$ of the approximate reduced system, given by (6), is a function of x, typical verification problems assume that initial conditions belong to a compact set S. The following result is therefore useful in those situations:

Proposition 3. *If $(\mathbb{R}^n, X)$ is approximately π_m-related to $(\mathbb{R}^m, Y)$, then for any compact set $S \subseteq \mathbb{R}^n$ there exists a $\delta \in \mathbb{R}^+$ such that for all $x \in S$ and all $t \in \mathbb{R}_0^+$ the following estimate holds:*

$$|\pi_m \circ \mathbf{x}(t, x) - \mathbf{y}(t, \pi_m(x))| \leq \delta. \tag{7}$$

Proof. Let $\delta = \max_{x \in C} \gamma(|\pi_k(x)|) + c$. The scalar δ is well defined since $\gamma(|\pi_k(\cdot)|) + c$ is a continuous map and C is compact. □

From a practical point of view, approximate reduction is only a useful concept if it admits characterizations that are simple to check. In order to derive such characterizations we need to review several notions of incremental stability.

4.1. Incremental stability

In this subsection we review two notions of incremental stability which will be fundamental in proving the main contribution of this paper. We follow [6] and [2].

Definition 5. A control system $(\mathbb{R}^n, \mathbb{R}^m, F)$ is said to be incrementally uniformly bounded-input-bounded-state stable (IUBIBSS) if there exist two class $\mathcal{K}_\infty$ functions γ_1 and γ_2 and a constant $d \in \mathbb{R}_0^+$ such that for each $x_1, x_2 \in \mathbb{R}^n$ and for each pair of smooth input curves $\mathbf{u}_1, \mathbf{u}_2 : \mathbb{R}_0^+ \to \mathbb{R}^m$ the following estimate holds for all $t \in \mathbb{R}_0^+$:

$$|\mathbf{x}_{\mathbf{u}_1}(t, x_1) - \mathbf{x}_{\mathbf{u}_2}(t, x_2)| \leq \gamma_1(|x_1 - x_2|) + \gamma_2(\|\mathbf{u}_1 - \mathbf{u}_2\|) + d. \tag{8}$$

A system is IUBIBSS when two different trajectories $\mathbf{x}_{\mathbf{u}_1}$ and $\mathbf{x}_{\mathbf{u}_2}$, starting at different but close initial conditions and associated with close but different input curves, will remain close for all time. In the linear case IUBIBSS turns out to be equivalent to stability but it is a distinct concept in the nonlinear case [6]. In general it is difficult to establish IUBIBSS directly. A sufficient condition is given by the existence of an IUBIBSS Lyapunov function. Note, however, that IUBIBSS only implies the existence of a IUBIBSS Lyapunov function with very weak regularity conditions [6].

Definition 6. A C^1 function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$ is said to be an IUBIBSS Lyapunov function for control system $(\mathbb{R}^n, \mathbb{R}^m, F)$ if there exist a $\xi \in \mathbb{R}^+$ and class $\mathcal{K}_\infty$ functions $\underline{\alpha}, \overline{\alpha}$, and μ such that for every $x_1, x_2 \in \mathbb{R}^n$ and $u_1, u_2 \in \mathbb{R}^m$ the following holds:

(1) $|x_1 - x_2| \geq \xi \implies \underline{\alpha}(|x_1 - x_2|) \leq V(x_1, x_2) \leq \overline{\alpha}(|x_1 - x_2|)$;
(2) $|x_1 - x_2| \geq \mu(|u_1 - u_2|) + \xi \implies \frac{\partial V}{\partial x_1} F(x_1, u_1) + \frac{\partial V}{\partial x_2} F(x_2, u_2) \leq 0$.

A stronger notion than IUBIBSS is incremental input-to-state stability.

Definition 7. A control system $(\mathbb{R}^n, \mathbb{R}^m, F)$ is said to be incrementally input-to-state stable (IISS) if there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}_\infty$ function γ such that for each $x_1, x_2 \in \mathbb{R}^n$ and for each pair of smooth curves $\mathbf{u}_1, \mathbf{u}_2 : \mathbb{R}_0^+ \to \mathbb{R}^m$ the following estimate holds for all $t \in \mathbb{R}_0^+$:

$$|\mathbf{x}_{\mathbf{u_1}}(t, x_1) - \mathbf{x}_{\mathbf{u_2}}(t, x_2)| \le \beta(|x_1 - x_2|, t) + \gamma(\|\mathbf{u}_1 - \mathbf{u}_2\|). \quad (9)$$

Since β is a decreasing function of t we immediately see that (9) implies (8) with $\gamma_1(r) = \beta(r, 0)$ and $\gamma_2(r) = \gamma(r)$, $r \in \mathbb{R}_0^+$. In addition to require trajectories to remain close if initial conditions and input curves are close, IISS requires the distance between trajectories to converge to zero over time. In the linear case IISS is equivalent to asymptotic stability but it is a distinct concept in the nonlinear case [2]. The notion of IISS is also implied by the existence of an IISS Lyapunov function. See [2] for a converse result when the inputs take values in a compact set.

Definition 8. A C^1 function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$ is said to be an IISS Lyapunov function for the control system $(\mathbb{R}^n, \mathbb{R}^m, F)$ if there exist class $\mathcal{K}_\infty$ functions $\underline{\alpha}, \overline{\alpha}, \alpha$, and μ such that for every $x_1, x_2 \in \mathbb{R}^n$ and $u_1, u_2 \in \mathbb{R}^m$ the following holds:

(1) $\underline{\alpha}(|x_1 - x_2|) \le V(x_1, x_2) \le \overline{\alpha}(|x_1 - x_2|)$;
(2) $|x_1 - x_2| \ge \mu(|u_1 - u_2|) \implies \frac{\partial V}{\partial x_1} F(x_1, u_1) + \frac{\partial V}{\partial x_2} F(x_2, u_2) \le -\alpha(|x_1 - x_2|)$.

4.2. Fiberwise stability

In addition to incremental stability we will also need a notion of partial practical stability. This notion will be used to ensure that the dynamics neglected in the approximate reduction process is well behaved.

Definition 9. A vector field $(\mathbb{R}^n, X)$ is said to be fibrewise practically stable with respect to π_k if there exist a class $\mathcal{K}_\infty$ function γ and a constant $c \in \mathbb{R}_0^+$ such that the following estimate holds for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}_0^+$:

$$\|\pi_k(\mathbf{x}(t, x))\| \le \gamma(|\pi_k(x)|) + c.$$

Fibrewise practical stability can be checked with the help of the following result:

Lemma 1. *A vector field $(\mathbb{R}^n, X)$ is fibrewise practically stable with respect to π_k if there exist two $\mathcal{K}_\infty$ functions, $\underline{\alpha}$ and $\overline{\alpha}$, a constant $d \in \mathbb{R}_0^+$, and a function $V : \mathbb{R}^n \to \mathbb{R}$ such that for every $x \in \mathbb{R}^n$ satisfying $|\pi_k(x)| \ge d$ we have:*

(1) $\underline{\alpha}(|\pi_k(x)|) \le V(x) \le \overline{\alpha}(|\pi_k(x)|)$,
(2) $\frac{\partial V}{\partial x} X(x) \le 0$.

4.3. Existence of approximate reductions

In this subsection we prove the main result providing sufficient conditions for the existence of approximate reductions.

Theorem 1. *Let $(\mathbb{R}^n, X)$ be a fibrewise practically stable vector field with respect to π_k and let $F = T\pi_m \cdot X : \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^m$, viewed as a control system with state space $\mathbb{R}^m$, be IUBIBSS. Then, the vector field $(\mathbb{R}^m, Y)$ defined by:*

$$Y(y) = T_{(y,0)}\pi_m \cdot X(y, 0) = F(y, 0)$$

for every $y \in \mathbb{R}^m$ is approximately π_m-related to $(\mathbb{R}^n, X)$.

Proof. By assumption, control system $(\mathbb{R}^m, \mathbb{R}^k, F = T\pi_m \circ X)$ is IUBIBSS. If we denote by $\mathbf{y}$ a trajectory of F we have:

$$|\mathbf{y}_{\mathbf{v}_1}(t, y_1) - \mathbf{y}_{\mathbf{v}_2}(t, y_2)| \le \gamma_1(|y_1 - y_2|) + \gamma_2(\|\mathbf{v}_1 - \mathbf{v}_2\|) + d.$$

In particular, we can take:

$$y_1 = y_2 = \pi_m(x), \qquad \mathbf{v}_1 = \pi_k \circ \mathbf{x}(\cdot, x), \qquad \mathbf{v}_2 = 0,$$

to get:

$$\begin{aligned} &|\pi_m \circ \mathbf{x}(t, x) - \mathbf{y}(t, \pi_m(x))| \\ &\quad = |\mathbf{y}_{\pi_k \circ \mathbf{x}(t,x)}(t, \pi_m(x)) - \mathbf{y}_0(t, \pi_m(x))| \\ &\quad = |\mathbf{y}_{\mathbf{v}_1}(t, \pi_m(x)) - \mathbf{y}_0(t, \pi_m(x))| \\ &\quad \le \gamma_2(\|\mathbf{v}_1\|) + d = \gamma_2(\|\pi_k \circ \mathbf{x}(\cdot, x)\|) + d. \end{aligned}$$

But it follows from fiberwise practical stability of X with respect to π_k that:

$$\|\pi_k \circ \mathbf{x}(\cdot, x)\| \le \gamma(|\pi_k(x)|) + c.$$

We thus have:

$$\begin{aligned} &|\pi_m \circ \mathbf{x}(t, x) - \mathbf{y}(t, \pi_m(x))| \le \gamma_2(\gamma(|\pi_k(x)|) + c) + d \\ &\quad \le \gamma_2(\lambda_1 \gamma(|\pi_k(x)|)) + \gamma_2(\lambda_2 c) + d, \end{aligned}$$

for some constants $\lambda_1, \lambda_2 \in \mathbb{R}_0^+$. This concludes the proof since $\gamma_2(\lambda_1 \gamma(|\cdot|))$ is a class $\mathcal{K}_\infty$ function and $\gamma_2(\lambda_2 c) + d \in \mathbb{R}_0^+$. □

Theorem 1 shows that sufficient conditions for approximate reduction can be given in terms of ISS-like Lyapunov functions and how reduced system can be constructed. Before illustrating Theorem 1 with several examples in the next section we present an important corollary.

Corollary 1. *Let $(\mathbb{R}^n, X)$ and $(\mathbb{R}^m, Y)$ be vector fields satisfying the assumptions of Theorem 1. Then, for any compact set $S \subseteq \mathbb{R}^n$ there exists a $\delta > 0$ such that for any $x \in S$ and $y \in \pi_m(S)$ the following estimate holds:*

$$|\pi_m \circ \mathbf{x}(t, x) - \mathbf{y}(t, y)| \le \delta.$$

Proof. Using the same proof as for Theorem 1, except picking $y_1 = \pi_m(x)$ and $y_2 = y$, it follows that:

$$\begin{aligned} &|\pi_m \circ \mathbf{x}(t, x) - \mathbf{y}(t, y)| \le \gamma_1(|\pi_m(x) - y|) \\ &\quad + \gamma_2(\lambda_1 \gamma(|\pi_k(x)|)) + \gamma_2(\lambda_2 c) + d. \end{aligned}$$

The bound δ is now given by:

$$\begin{aligned} \delta = &\max_{(x,y) \in S \times \pi_m(S)} \gamma_1(|\pi_m(x) - y|) \\ &\quad + \gamma_2(\lambda_1 \gamma(|\pi_k(x)|)) + \gamma_2(\lambda_2 c) + d \end{aligned}$$

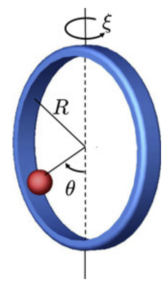

Fig. 1. Ball in a rotating hoop.

which is well defined since $S \times \pi_m(S)$ is compact. □

5. Examples

In this section, we consider examples that illustrate the usefulness of approximate reduction.

5.1. Ball in the hoop

As a first example we consider the ball in a rotating hoop with friction, as described in Chapter 2 of [9] and displayed in Fig. 1. For this example there are the following parameters: m — mass of the ball, R — radius of the hoop, g — acceleration due to gravity, μ — friction constant for the ball, and ξ — angular velocity for the ball. The equations of motion are given by:

$$\dot{\omega} = -\frac{\mu}{m}\omega + \xi^2 \sin\theta\cos\theta - \frac{g}{R}\sin\theta$$
$$\dot{\theta} = \omega \qquad (10)$$

where θ is the angular position of the ball and ω is its angular velocity.

If $\pi_\omega : \mathbb{R}^2 \to \mathbb{R}$ is the projection $\pi_\omega(\omega, \theta) = \omega$, then according to Proposition 2 there exists no vector field Y on $\mathbb{R}$ which is π_ω-related to X (as defined by (10)). However, we will show that $Y(\omega) = T_{(\omega,0)}\pi_\omega \cdot X(\omega, 0)$ is approximate π_ω-related to X.

In applying Theorem 1 we follow three steps:

(1) We show that X is forward complete;
(2) We show that X is fiberwise practically stable;
(3) We show that $F = T\pi_m \cdot X : \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^m$ is IUBIBSS;
(4) We construct the reduced model Y.

5.1.1. Step 1: Forward completeness

We use:

$$V = \frac{1}{2}mR^2\omega^2 + mgR(1 - \cos\theta) - \frac{1}{2}mR^2\xi^2\sin^2\theta$$

as a Lyapunov function to show that (10) is stable. Note that $V(\omega, \theta) = 0$ for $(\omega, \theta) = (0, 0)$ and $V(\omega, \theta) > 0$ for $(\omega, \theta) \neq (0, 0)$ provided that $R\xi^2 < g$, which we assume. Computing the time derivative of V we obtain:

$$\dot{V} = -\mu R^2\omega^2 \leq 0,$$

thus showing stability of (10) and forward completeness.

5.1.2. Step 2: Fiberwise practical stability

We now consider a compact set C invariant under the dynamics and restrict our analysis to initial conditions in this set. Such a set can be constructed, for example, by taking $\{x \in \mathbb{R}^2 \mid V(x) \leq c\}$ for some positive constant c. Note that stability of (10) implies fibrewise stability on C since $\pi_m(C)$ is compact.

5.1.3. Step 3: IUBIBSS

We will show that:

$$T_{(\omega,\theta)}\pi_\omega \cdot X(\omega, \theta) = -\frac{\mu}{m}\omega + \xi^2\sin\theta\cos\theta - \frac{g}{R}\sin\theta$$

is IUBIBSS on C with θ seen as an input by proving the stronger property of IISS. Consider the function:

$$U = \frac{1}{2}(\omega_1 - \omega_2)^2.$$

Its time derivative is given by:

$$\begin{aligned}
\dot{U} &= (\omega_1 - \omega_2)\Big[-\frac{\mu}{m}(\omega_1 - \omega_2) + \xi^2\sin\theta_1\cos\theta_1 \\
&\quad - \frac{g}{R}\sin\theta_1 - \xi^2\sin\theta_2\cos\theta_2 + \frac{g}{R}\sin\theta_2\Big] \\
&\leq -\frac{\mu}{m}(\omega_1 - \omega_2)^2 + |\omega_1 - \omega_2|\,\Big|\xi^2\sin\theta_1\cos\theta_1 \\
&\quad - \frac{g}{R}\sin\theta_1 - \xi^2\sin\theta_2\cos\theta_2 + \frac{g}{R}\sin\theta_2\Big| \\
&\leq -\frac{\mu}{m}(\omega_1 - \omega_2)^2 + |\omega_1 - \omega_2|L|\theta_1 - \theta_2| \\
&= -\frac{\mu}{2m}(\omega_1 - \omega_2)^2 \\
&\quad + \Big(-\frac{\mu}{2m}(\omega_1 - \omega_2)^2 + |\omega_1 - \omega_2|L|\theta_1 - \theta_2|\Big), \qquad (11)
\end{aligned}$$

where the second inequality follows from the fact that $\xi^2\sin\theta\cos\theta - \frac{g}{R}\sin\theta$ is a smooth function defined on the compact set $\pi_\theta(C)$ and is thus globally Lipschitz on $\pi_\theta(C)$ (since its derivative is continuous and thus bounded on any compact convex set containing $\pi_\theta(C)$) with Lipschitz constant L. We now note that the condition:

$$|\omega_1 - \omega_2| > \frac{2mL}{\mu}|\theta_1 - \theta_2|$$

makes the second term in (11) negative from which we conclude the following implication:

$$|\omega_1 - \omega_2| > \frac{2mL}{\mu}|\theta_1 - \theta_2| \Longrightarrow \dot{U} \leq -\frac{\mu}{2m}(\omega_1 - \omega_2)^2$$

showing that U is an IISS Lyapunov function for (10) and thus concluding IUBIBSS.

5.1.4. Step 4: Construction of the reduced model

According to Theorem 1 the approximate reduction of (10) is given by:

$$\dot{\omega} = -\frac{\mu}{m}\omega.$$

Projected trajectories of the full-order system as compared with trajectories of the reduced system can be seen in Fig. 2; here

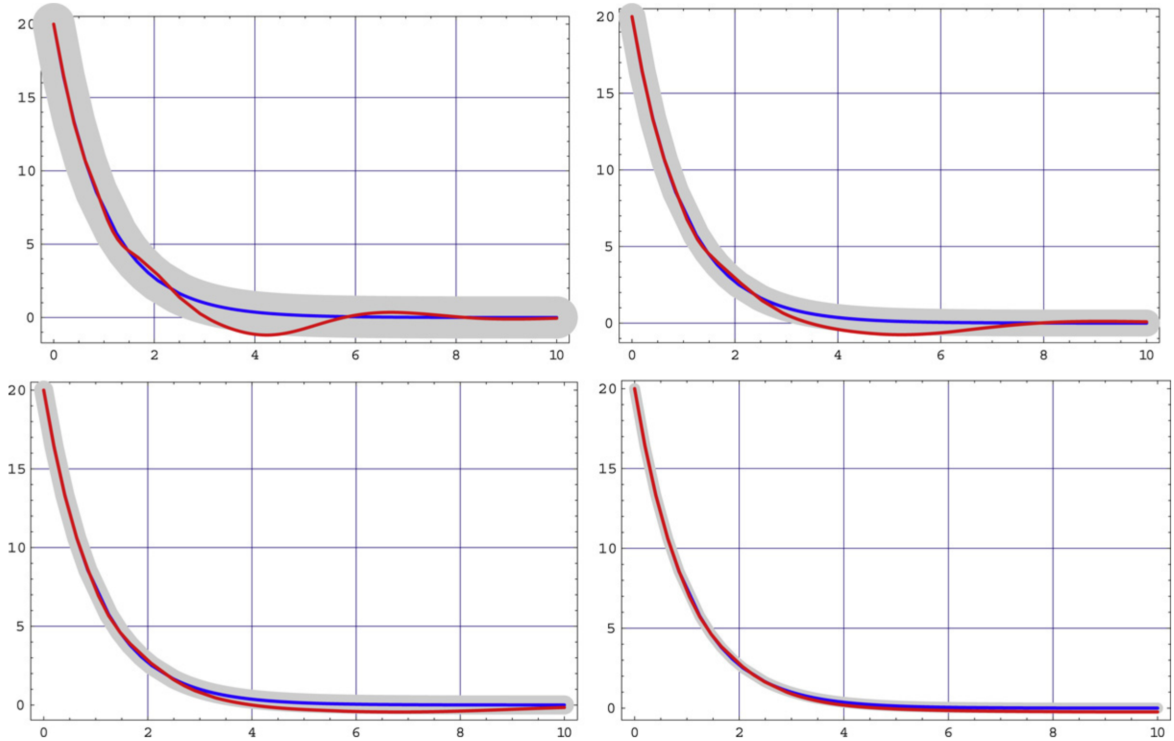

Fig. 2. A trajectory of the full order system (red) vs. a trajectory for the reduced system (blue) for $R = 5, 10, 20, 40$ (from left to right and top to bottom, respectively). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

$\mu = m = 1$ and $\xi = 0.1$. Note that as $R \to \infty$, the reduced system converges to the full-order system (or the full-order system effectively becomes decoupled).

5.2. Pendulum on a cart

We now consider a pendulum attached to a cart which is mounted to a spring (see Fig. 3). For this example, there are the following parameters: M — mass of the cart, m — mass of the pendulum, R — length of the rod, k — spring stiffness, g — acceleration due to gravity, d — friction constant for the cart, and b — friction constant for the pendulum. The equations of motion are given by:

$$\begin{aligned}
\dot{x} &= v \\
\dot{\theta} &= \omega \\
\dot{v} &= \frac{1}{M + m\sin^2\theta} \\
&\quad \times \left(mR\omega^2 \sin\theta + mg\sin\theta\cos\theta - kx - dv + \frac{b}{R}\cos\theta \right) \\
\dot{\omega} &= \frac{1}{R(M + m\sin^2\theta)} \Big(-mR\omega^2 \sin\theta\cos\theta \\
&\quad - (m + M)g\sin\theta + kx\cos\theta + dv\cos\theta \\
&\quad - \left(1 + \frac{M}{m}\right)\frac{b}{R}\omega \Big)
\end{aligned} \tag{12}$$

where x is the position of the cart, v its velocity, θ is the angular position of the pendulum and ω its angular velocity.

If $\pi_{(x,v)} : \mathbb{R}^4 \to \mathbb{R}^2$ is the projection: $\pi_{(x,v)}(x, \theta, v, \omega) = (x, v)$ and X is the vector field as defined in (12), the goal is to reduce X to $\mathbb{R}^2$ by eliminating the θ and ω variables.

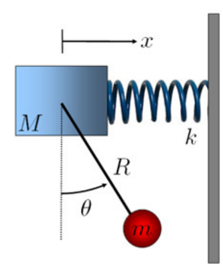

Fig. 3. A graphical representation of the pendulum on a cart mounted on a spring.

5.2.1. Steps 1 and 2: Forward completeness and fibrewise practical stability

Forward completeness and fibrewise practical stability of (12) are prove like in the previous example by noting that X is Hamiltonian for $d = b = 0$ and using the Hamiltonian as a Lyapunov function V.

5.2.2. Step 3: IUBIBSS

Control system $F = T\pi_{x,v} \cdot X(x, \theta, v, w)$, in which θ and ω are regarded as inputs, is given by:

$$\begin{aligned}
F\left((x, v), (\theta, \omega)\right) &= T\pi_{(x,v)} \cdot X(x, \theta, v, \omega) \\
&= \frac{1}{M + m\sin^2\theta} \Big(mR\omega^2 \sin\theta - kx \\
&\quad + mg\sin\theta\cos\theta - dv + \frac{b}{R}\cos\theta \Big).
\end{aligned} \tag{13}$$

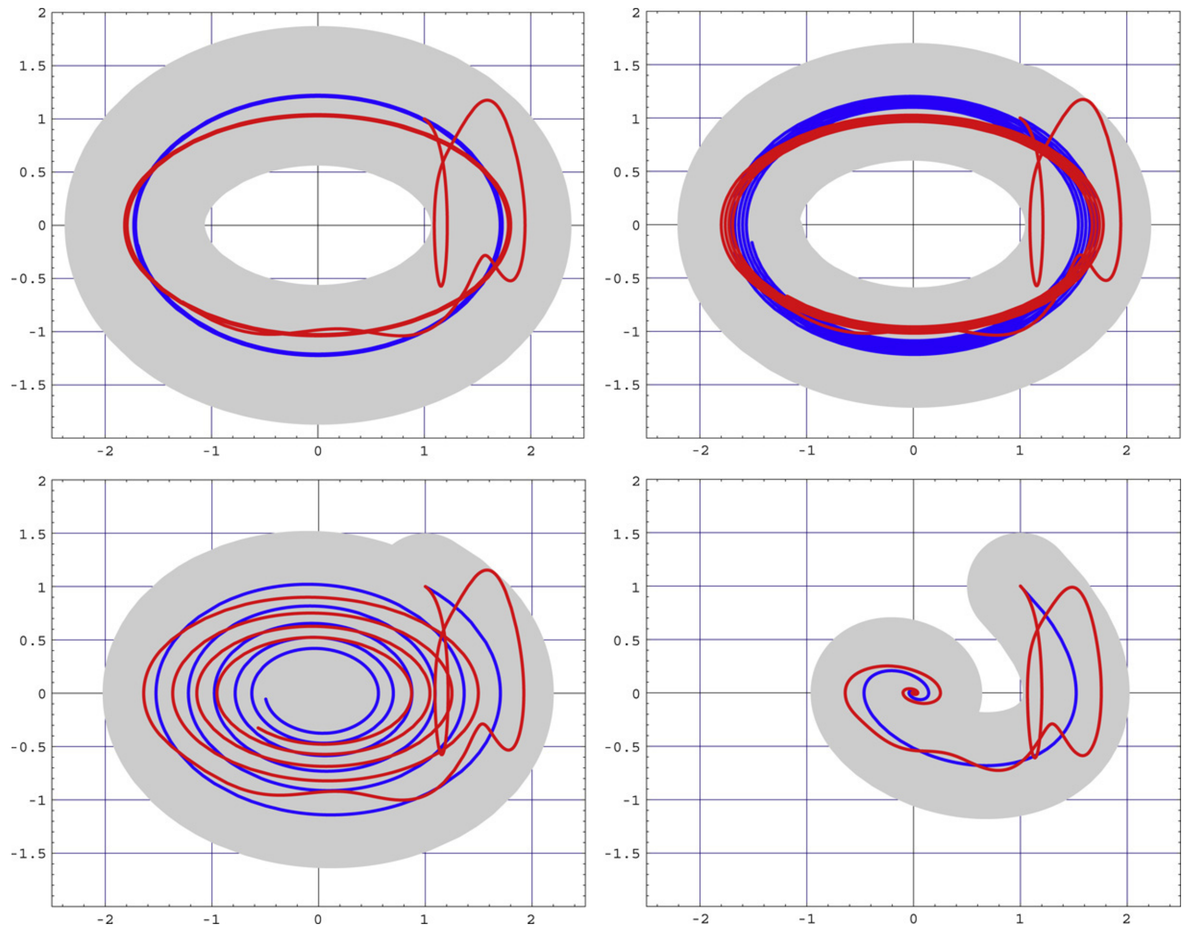

Fig. 4. A projected trajectory of the full-order system (red) and a trajectory for the reduced system (blue) for $d = 0.001, 0.01, 0.1, 1$ (from left to right and top to bottom, respectively). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

To show that F is IUBIBSS we first rewrite (13) in the form:

$$F\left((x, v), (\theta, \omega)\right) = \frac{1}{M+m}\left(mR\omega^2 \sin\theta - kx - dv - mR\dot{\omega}\cos\theta\right)$$

and consider the following IISS candidate Lyapunov function:

$$U = \frac{1}{2(m+M)}(x_1 - x_2)^2 + \frac{1}{2}(v_1 - v_2)^2.$$

Its time derivative is given by:

$$\dot{U} = -\frac{d}{m+M}(v_1 - v_2)^2 + \frac{mR}{m+M} \times \left(\omega_1^2 \sin\theta_1 - \dot{\omega}_1 \cos\theta_1 - \omega_2^2 \sin\theta_2 + \dot{\omega}_2 \cos\theta_2\right) \times (v_1 - v_2).$$

Using an argument similar to the one used for the previous example, we conclude that:

$$|v_1 - v_2| \geq \frac{2mRL}{d}\left|(\theta_1, \omega_1, \dot{\omega}_1) - (\theta_2, \omega_2, \dot{\omega}_2)\right|,$$

with L the Lipschitz constant of the function $\omega^2 \sin\theta - \dot{\omega}\cos\theta$, implies:

$$\dot{U} \leq -\frac{d}{2(m+M)}(v_1 - v_2)^2,$$

thus showing that X is IISS and in particular also IUBIBSS.

5.2.3. Step 4: Construction of the reduced model

The reduced model $Y(x, v)$ is given by:

$$\begin{pmatrix}\dot{x}\\ \dot{v}\end{pmatrix} = Y(x, v) = T_{(x,0,v,0)}\pi_{(x,v)} \cdot X(x, 0, v, 0) = \begin{pmatrix} v \\ -\frac{1}{M}(dv + kx)\end{pmatrix}.$$

In order to illustrate some of the interesting implications of approximate reduction, we compare the reduced system, Y, and the full-order system, X, in the case when $R = m = k = b = 1$ and $M = 2$. It follows that the equations of motion for the reduced system are given by the linear system:

$$\begin{pmatrix}\dot{x}\\ \dot{v}\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & d\end{pmatrix}\begin{pmatrix} x \\ v\end{pmatrix},$$

so we can completely characterize the dynamics of the reduced system: every solution spirals into the origin. This is in stark contrast to the dynamics of X (see (12)) which are very complex. The fact that X and Y are approximately related, and more specifically Theorem 1, allows us to understand the dynamics of X through the simple dynamics of Y. To be more specific, because the distance between the projected trajectories of X and the trajectories of Y is bounded, we know that the projected trajectories of X are "essentially" be spirals. Moreover, the friction constant d will directly affect the rate of convergence of these spirals. Examples of this can be seen in Fig. 4 where d is varied to affect the convergence of the reduced system, and hence the full order system.

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