Coordination of Multiple Autonomous Vehicles

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Abstract—**We analyze the coordinated motion of a group of nonholonomic vehicles that are controlled in a distributed fashion to exhibit flocking behavior. This behavior emerges from aggregating the control actions of all group members; it is not imposed by some centralized control scheme. Each vehicle is locally controlled by a combination of a potential field force and an alignment force. The former control component ensures collision avoidance and attraction towards the group, while the latter steers each vehicle to the average heading of its 'neighbors'. Eventually all vehicles attain a common heading and move in tight formation while avoiding collisions.**

I. Introduction

Recent technological advances offered more efficient computation and less expensive communication. The ability to compute locally and share information has facilitated the development of new multi-agent systems. Such type of systems promise increased performance, efficiency and robustness, at a fraction of the cost compared to their centralized counterparts, utilizing distributed coordination sensing and actuation. The question that now arises is how to achieve the desired level of coordination in multi-agent systems.

Nature is abundant in marvelous examples of coordinated behavior. Across the scale, from biochemical cellular networks up to ant colonies, schools of fish, flocks of birds and herds of land animals, one can find systems that exhibit astonishingly efficient and robust coordination schemes [1, 23, 6]. At the same time several researchers in the area of statistical physics and complexity theory have addressed flocking and schooling behavior in the context of non-equilibrium phenomena in many-degree-of-freedom dynamical systems and self organization in systems of self-propelled particles [22, 21, 13, 11, 17]. Similar problems have become a major thrust in systems and control theory, in the context of cooperative control, distributed control of multiple vehicles and formation control; see for example [10, 3, 15, 5, 12, 7, 18, 9, 14, 4]. The main goal of the above papers is to develop a decentralized control strategy so that a global objective, such as a tight formation with fixed pair-wise inter vehicle distances, is achieved.

In 1986 Craig Reynolds [16] made a computer model of coordinated animal motion such as bird flocks and fish schools. He called the generic simulated flocking creatures "boids". The basic flocking model consists of three simple steering behaviors which describe how an individual boid maneuvers based on the positions and velocities its nearby flockmates: separation, alignment, and cohesion. In 1995, a similar model was proposed by Vicsek *et al.* [22]. Under an alignment rule, a spontaneous development of coherent collective motion is observed, resulting in the headings of all agents to converge to a common value. A proof of convergence for Vicsek's model (in the noise-free case) was given in [9].

In this paper provide a system theoretic justification for an instance of the flocking phenomenon observed in [16]. In our flock model, we consider dynamic nonholonomic systems steered in a decentralized fashion. We show that all agents headings converge to the same value, velocities will eventually become the same and pairwise distances will converge. Our analysis makes use of Lyapunov stability and algebraic graph theory. While the proof techniques are totally different from those in [9], the end result is similar, suggesting that addition of cohesion and separation forces in addition to alignment as well as addition of dynamics, does not affect the stability of the flocking motion.

The paper is organized as follows: In Section II we describe the dynamics of each vehicle in the group and introduce its control law. Section III shows how this control law gives rise to flocking behavior with simultaneous collision avoidance. Numerical simulations verifying the stability results are presented in Section IV and the paper concludes with a summary of its contributions in Section V.

II. Flocking Control

This Section presents the model and the control strategy for the group. The control laws assume a certain interconnection topology on the group of vehicles, through which state information is exchanged (either by means of sensing or communication) for the purposes of coordination. The analysis in this paper assumes that the interconnection topology is time invariant. An instance of the problem in the time varying case with holonomic agents is treated in [20], while the equivalent time invariant case is investigated in [19].

Consider a group of N vehicles, moving on the plane according to the following dynamics:

$$\dot{x}_i = v_i \cos\theta_i \quad (1a)$$
$$\dot{y}_i = v_i \sin\theta_i \quad (1b)$$
$$\dot{\theta}_i = \omega_i \quad (1c)$$
$$\dot{v}_i = a_i, \quad (1d)$$

for $i = 1, \ldots, N$, where $r_i = (x_i, y_i)^T$ is the position vector of vehicle i, θ_i its orientation (Figure 1), v_i its translational speed and a_i, ω_i its control inputs. The relative positions between the vehicles are denoted $r_{ij} \triangleq r_i - r_j$.

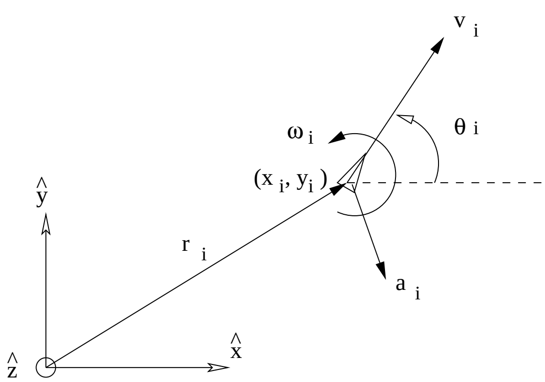

Fig. 1. Control forces acting on vehicle i.

Let us now assume a fixed control interconnection topology on the group of vehicles. Interconnections express control dependences; for instance, if vehicle i is interconnected to j and k, this means that the control inputs of i depend on the states of vehicles j and k. We will call the set of vehicles that are interconnected with i, the *neighbors* of i. The neighboring relations are represented by means of an undirected graph:

Definition II.1 (Neighboring graph) *The neighboring graph, $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{W}\}$, is a labeled undirected graph consisting of*

- *a set of vertices, $\mathcal{V} = \{n_1, \ldots, n_N\}$, indexed by the vehicles in the group,*
- *a set of edges, $\mathcal{E} = \{(n_i, n_j) \mid n_i \sim n_j \text{ for } n_i, n_j \in \mathcal{V}\}$, containing unordered pairs of vertices representing neighboring relations, and*
- *a set of labels, $\mathcal{W}$, indexed by the edges, and a map associating each edge with a label $w \in \mathcal{W}$, equal to the product of the speeds of the agents corresponding to the adjacent vertices, $w_{ij} = |v_i||v_j|$.*

For every pair of neighboring vehicles, $(i, j) \in \mathcal{E}$ we consider an artificial potential function V_{ij} that depends on the distance between i and j. We do not require a particular form for V_{ij}; any function will do, provided that it depends only on $\|r_{ij}\|$. From the definition of r_{ij} it follows that V_{ij} is symmetric with respect to r_i and r_j. As example of such function, used also in our simulation examples, is the following:

$$V_{ij}(\|r_{ij}\|) = \frac{1}{\|r_{ij}\|^2} + \log \|r_{ij}\|^2 .$$

The graph of this function is given in Figure 2. Let us now define the potential energy for vehicle i as

$$V_i \triangleq \sum_{j \sim i} V_{ij}(\|r_{ij}\|).$$

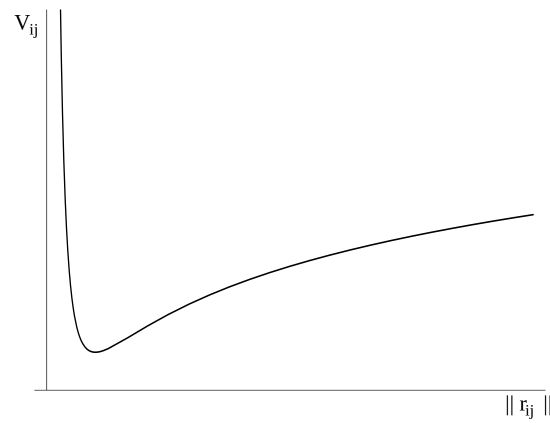

Fig. 2. The artificial potential between two vehicles.

Then the control inputs for vehicle i can be defined in terms of the gradient of the potential function and the relative headings and distances between vehicle i and its neighbors,

$$a_i = -(\nabla_{r_i} V_i)_x \cos\theta_i - (\nabla_{r_i} V_i)_y \sin\theta_i \quad (2a)$$
$$\omega_i = -k \sum_{j \sim i} |v_i||v_j|(\theta_i - \theta_j) + \frac{(\nabla_{r_i} V_i)_x \sin\theta_i - (\nabla_{r_i} V_i)_y \cos\theta_i}{|v_i|} \quad (2b)$$

with k being a constant positive parameter (control gain).

The control input (2a)-(2b), can be thought of as a force f_i acting on each mobile agent. This force is the resultant of a potential force and an alignment force:

$$f_i = -\nabla_{r_i} V_i + -k\hat{z} \times \dot{r}_i \sum_{j \sim i} |v_i||v_j|(\theta_i - \theta_j)$$

The second term, also known in literature as a *gyroscopic force* [2], is not affecting its kinetic energy, since it acts along a direction normal to the velocity vector of agent i.

III. Closed Loop Stability

In this Section we will show that under the control law (2a)-(2b) the group of nonholonomic vehicles forms a tight flock which moves uniformly, while avoiding collisions between the vehicles. The proof technique is based on a combination of mechanics, graph theory and Lyapunov stability.

Let us assume an arbitrary orientation σ on $\mathcal{G}$, and denote the oriented neighboring graph $\mathcal{G}^\sigma$. The choice of orientation is not important and is simply reflects the orientation chosen for the relative position vectors r_{ij} between the mobile agents. Let B be the incidence matrix of $\mathcal{G}^\sigma$, defined as a matrix with entries $\{-1, 0, 1\}$ with rows indexed by the set of vertices and columns indexed by the set of edges. The (i, j) entry of B is -1 if for vertex i edge j is outgoing, 1 if j is incoming and 0 if i is not adjacent to edge j.

Define the total mechanical energy of the group as follows:

$$V_t = \frac{1}{2}\sum_{j\sim i} V_{ij} + \frac{1}{2}v_i^2. \tag{3}$$

Due to V_{ij} being symmetric with respect to r_{ij}, and the fact that $r_{ij} = -r_{ji}$, the partial derivatives of V_{ij} satisfy:

$$\frac{\partial V_{ij}}{\partial r_{ij}} = \frac{\partial V_{ij}}{\partial r_i} = -\frac{\partial V_{ij}}{\partial r_j}, \tag{4}$$

It can easily be verified that the action of gyroscopic (alignment) forces do not affect the level of (3). What they do, in fact, is to transfer mechanical energy from each pair of neighbors to the kinetic energy of the "centroid" of the group, a position defined as:

$$r_c \triangleq \frac{1}{N}\sum_{i=1}^{N} r_i$$

Let the kinetic energy of the centroid be expressed by the function

$$V_c \triangleq \frac{1}{2}\left(\sum_{i=1}^{N}\dot{r}_i\right)^T\left(\sum_{i=1}^{N}\dot{r}_i\right)$$

In what follows we will show that this function increases monotonically under the control scheme (2a)-(2b). For stability purposes we will consider the difference between the total mechanical energy V_t, which is a conserved quantity, and the centroid kinetic energy V_c. Since V_t is conserved, we can just as well consider its value at initial time, defining this difference as:

$$W \triangleq V_t\Big|_{t=0} - V_c \tag{5}$$

To show monotonic decrease of W, we need the system to live inside a certain compact set Ω. Let θ denote the stack vector of all θ_i, r denote the stack vector of all r_i and allow an abuse of notation by denoting the stack vector of the sinuses of all θ_i by $\sin\theta$. Define Ω as follows:

$$\begin{aligned}\Omega \triangleq \{(r_{ij}, v_i, \theta_i) \mid V_t \le c,\\ \text{and } \sin(B^T\theta)^T(B^TB)B^T\theta \ge 0,\ i,j = 1,\dots N\},\end{aligned}$$

Set Ω is nonempty for a sufficiently large choice of c, and closed by continuity of V_t. Boundedness of Ω follows from the fact that $V_t \le c$ implies boundedness of all V_{ij}, and since V_{ij} increases monotonically with r_{ij}, all r_{ij} have to be bounded as well. Bounds for v_i can be found similarly and θ_i is always in the interval $[-\pi, \pi]$. Thus, Ω is compact. The argument that follows ensures that in addition, Ω can be made positively invariant.

The condition $\sin(B^T\theta)^T(B^TB)B^T\theta \ge 0$ can be satisfied in finite time, by an appropriately large choice of control gain k. This can be seen from (2b), if we collect the angular velocity inputs in one expression:

$$\dot{\theta} = -kL_w\theta - F$$

where $F^T = [\cdots\ \nabla_{r_i}V_i(\hat{z}\times\dot{r}_i)\ \cdots]$ and L_w is a weighted Laplacian of the neighboring graph $\mathcal{G}$, defined as:

$$L_w \triangleq BD_wB^T, \text{ with } D_w = \text{diag}(|v_i||v_j|)$$

The definition of the Laplacian is independent of the orientation chosen for the graph. For the Laplacian, it is known [8] that it has as many zero eigenvalues as the number of connected components in the graph and that in the case of a connected graph, the eigenvector associated with the zero eigenvalues is the N-dimensional vector of ones, $\mathbf{1}_N$.

It is clear that for $r_{ij} \ne 0$, there will always be an upper bound on $\|F\|$. On the other hand, for a connected graph $\mathcal{G}$, L_w has a single zero eigenvalue with corresponding eigenvector $\mathbf{1}_N$ and a series of $N-1$ real positive eigenvalues. Thus, a sufficiently large k ensures that θ approaches an arbitrarily small neighborhood of $\mu\mathbf{1}_N$, for some constant number $\mu \in [-\pi, \pi]$. Therefore $B^T\theta$ can be made sufficiently small in finite time. With $B^T\theta$ living in a small region around the origin, $\sin(B^T\theta) \approx B^T\theta$ and given the positive semi-definiteness of B^TB, we can have $\sin(B^T\theta)^TB^TB(B^T\theta) \ge 0$. This means that even if the system starts with initial conditions outside Ω, we can steer it inside Ω in finite time.

The proposition that follows establishes the stability properties of the group motion, under the control laws (2a)-(2b). Once inside the set Ω, the group flocks, in the sense that it forms a tight formation where all members move in the same direction, distances between the agents become fixed and velocities all converge to the same values.

Proposition III.1 *Consider a system of N mobile agents with dynamics* (1) *steered by control laws* (2a)-(2b)*. Then, for a sufficiently large gain k and for initial conditions in Ω, all headings will converge to a common direction and inter-vehicle distances approach values that correspond to a minimum of the group artificial potential $\sum_{i=1}^{N} V_i$.*

Proof: Taking the time derivative of function W:

$$\dot{W} = -\dot{r}_c^T \ddot{r}_c = -\sum_{i=1}^{N} \dot{r}_i^T \cdot \sum_{i=1}^{N} \ddot{r}_i$$
$$= \sum_{i=1}^{N} \dot{r}_i^T \cdot \sum_{i=1}^{N} \left(\sum_{j\sim i} \nabla_{r_i} V_{ij} + k \sum_{j\sim i} \frac{\theta_i - \theta_j}{\|r_{ij}\|^2} (\hat{z} \times \dot{r}_i) \right)$$

The above can be written more compactly using matrix algebraic notation. where $\otimes$ denotes the Kronecker matrix product, and $\circ$ denotes the Hadamard product this is written as

$$\begin{aligned}\dot{W} =& [(\mathbf{1}_N^T \otimes I_3)\dot{r}]^T \cdot [(\mathbf{1}_N^T \otimes I_3)\big((B \otimes I_3)\tfrac{\partial V_{ij}}{\partial \|r_{ij}\|}\hat{r}_{ij} \\ &+ k(L_w\theta \otimes \mathbf{1}_3) \circ (\hat{z} \times \dot{r})\big)] \\ =& [(\mathbf{1}_N^T \otimes I_3)\dot{r}]^T \cdot [(\mathbf{1}_N^T \otimes I_3)((B \otimes I_3)\tfrac{\partial V_{ij}}{\partial \|r_{ij}\|}\hat{r}_{ij}) \\ &+ (\mathbf{1}_N^T \otimes I_3)\big(k(L_w\theta \otimes \mathbf{1}_3) \circ (\hat{z} \times \dot{r})\big)] \\ =& [(\mathbf{1}_N^T \otimes I_3)\dot{r}]^T \cdot [((\mathbf{1}_N^T B) \otimes I_3)\tfrac{\partial V_{ij}}{\partial \|r_{ij}\|}\hat{r}_{ij} \\ &+ (\mathbf{1}_N^T \otimes I_3)\big(k(L_w\theta \otimes \mathbf{1}_3) \circ (\hat{z} \times \dot{r})\big)] \\ =& [(\mathbf{1}_N^T \otimes I_3)\dot{r}]^T \cdot [(\mathbf{1}_N^T \otimes I_3)\big(k(L_w\theta \otimes \mathbf{1}_3) \circ (\hat{z} \times \dot{r})\big)] \\ =& [(\mathbf{1}_N^T \otimes I_3)\dot{r}]^T \cdot [(\mathbf{1}_N^T \otimes I_3)\big(\mathrm{diag}(k(L_w\theta) \otimes \mathbf{1}_3)(\hat{z} \times \dot{r})\big)].\end{aligned}$$

This derivative is rewritten using $\bar{\theta}_i$ to denote the i^{th} element of $L_w\theta$, as

$$\begin{aligned}\dot{W} =& ([\,\dot{r}_1 \cdots \dot{r}_N\,]\mathbf{1}_N)^T \cdot k([\,\bar{\theta}_1\hat{z}\times\dot{r}_1 \cdots \bar{\theta}_N\hat{z}\times\dot{r}_N\,]\mathbf{1}_N) \\ =& k\sum_{i=1}^{N} \dot{r}_i^T \cdot \sum_{j=1}^{N} \bar{\theta}_j \hat{z} \times \dot{r}_j = k \sum_{j=1}^{N}\sum_{i=1}^{N} \bar{\theta}_j \hat{z} \cdot (\dot{r}_i \times \dot{r}_j) \\ =& -k \sum_{j=1}^{N}\sum_{i=1}^{N} \bar{\theta}_j \hat{z} \cdot (\dot{r}_j \times \dot{r}_i) \\ =& -k \sum_{i,j=1}^{N} (\bar{\theta}_j - \bar{\theta}_i)\hat{z} \cdot (\dot{r}_j \times \dot{r}_i) \\ =& -k \sum_{i,j=1}^{N} (\bar{\theta}_j - \bar{\theta}_i)|v_j||v_i| \sin(\theta_j - \theta_i).\end{aligned}$$

For pairs (i, j) not belonging in the edge set of the neighboring graph, we have :

$$\begin{aligned}&|v_j||v_i|(\bar{\theta}_j - \bar{\theta}_i)\sin(\theta_j - \theta_i) \\ &\quad = |v_j||v_i|(\theta_j - \theta_i)\sin(\theta_j - \theta_i) \geq 0, \ \text{for } (i, j) \notin \mathcal{E}\end{aligned}$$

For pairs (i, j) belonging to the edge set of $\mathcal{G}$, we can write:

$$\sum_{(i,j)\in\mathcal{E}} |v_j||v_i| \sin(\theta_j - \theta_i)(\bar{\theta}_j - \bar{\theta}_i) = \sin(B^T\theta)^T D_w B^T B D_w B^T \theta.$$

This bilinear form is positive semi-definite in Ω. Then,

$$\begin{aligned}\dot{W} = &-k \sum_{(i,j)\notin\mathcal{E}} \|\dot{r}_j\|\,\|\dot{r}_i\|\,(\theta_j - \theta_i)\sin(\theta_j - \theta_i) \\ &- k\sin(B^T\theta)^T D_w B^T B D_w B^T \theta \leq 0\end{aligned}$$

Applying LaSalle's invariant principle on (1) in Ω we conclude that all trajectories converge to the largest invariant set in $S = \{(r_{ij}, v_i, \theta_i) \mid \dot{W} = 0,\ i, j = 1, \ldots, N\}$. Provided at least one pair of neighboring agents is moving, $\dot{W} = 0$ requires $\theta_i - \theta_j = 0$, for all $i, j \in \mathcal{V}$. Even if all agents are not moving, they cannot continue to do so, unless they are located in configurations of minimal potential energy $\sum_{i=1}^{N} V_i$.

In an invariant set in S, (2b) suggests that in addition to $\theta = \mu\mathbf{1}_N$, there must also be $\nabla_{r_i} V_i(\hat{z} \times \dot{r}_i) = 0$ for all $i \in \mathcal{V}$. The latter only happens when for all $i \in \mathcal{V}$, we either have $\nabla_{r_i} V_i = 0$ or $\nabla_{r_i} V_i$ being parallel to $\dot{r}_i$. In other words, the invariant set consists of configurations where all agents have a common heading and are either moving along the same straight line, or that have reached a configuration of local minimum of $\sum_{i=1}^{N} V_i$.

■

Collision avoidance is guaranteed since in all configurations where $r_{ij} = 0$ for some $i, j \in \{1, \ldots, N\}$, the function V_t tends to infinity implying that these configurations lay in the exterior of any Ω with c bounded.

IV. Simulations

In this Section we verify numerically the stability results of Section III. The group is consisted of five autonomous agents with dynamics described by (1). The initial conditions (positions and velocities) were generated randomly within a ball of radius $R_0 = 5$[m]. Figures 3-8 show snapshots of the group motion during a simulation time period of 15 seconds. In these Figures the position of the agents are represented by (red) dots, connected to each other by (blue) line segments which represent the neighboring relations. The dashed line trails left behind by the dots correspond to agent paths. Figures 3-8 show how the headings of all agents asymptotically approach a common direction. The asymptotic convergent behavior of the relative heading differences is depicted in Figure 9. Figure 10 gives an enlarged picture of the shape of the group after the period of 15 simulation seconds. The agents tend to attain positions on the vertices of an inscribed polygon with 5 faces, which corresponds to a local minimum of the group potential function.

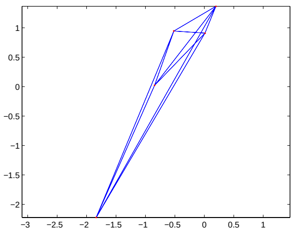

Fig. 3. Initial state.

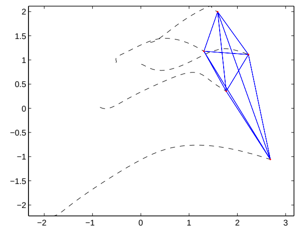

Fig. 4. Sim. time $t = 3$ sec.

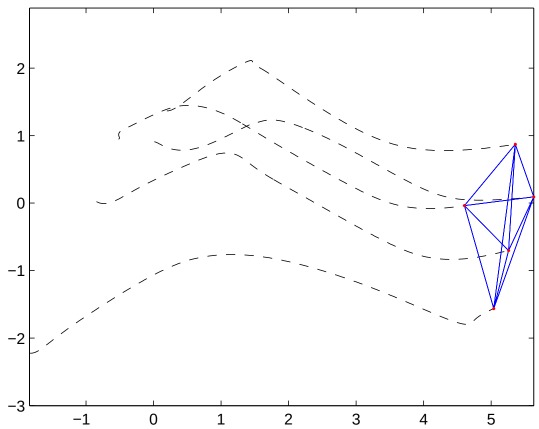

Fig. 5. Sim. time $t = 6$ sec.

V. Conclusions

In this paper we proposed a set of control laws that give rise to stable flocking motion for a group of nonholonomic vehicles capable of sharing state information, through a fixed control interconnection topology. In this way, we theoretically explained the flocking behavior observed in the animation models of [16]. The proof is based on the mechanics of a system of particles and the connectivity properties of the graph of inter-vehicle interconnections. Further research efforts are directed towards characterizing the robustness properties of this control design and the relation between these robustness properties and the graph algebraic properties of the interconnection topology of the group.

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References

[1] C. M. Breder. Equations descriptive of fish schools and other animal aggergations. *Ecology*, 35:361–370, 1954.

[2] D. E. Chang, A. M. Bloch, N. E. Leonard, J. E. Marsden, and C. A. Woosley. The equivalence of con-

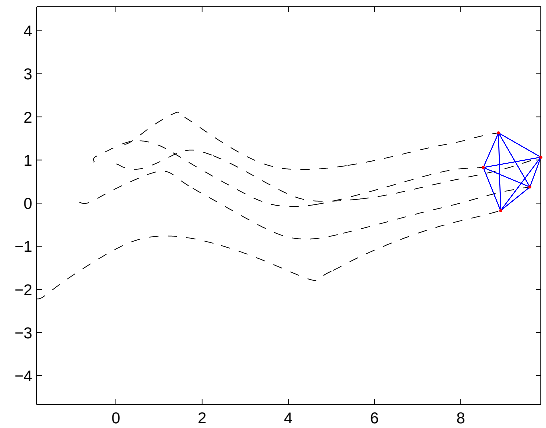

Fig. 6. Sim. time $t = 9$ sec

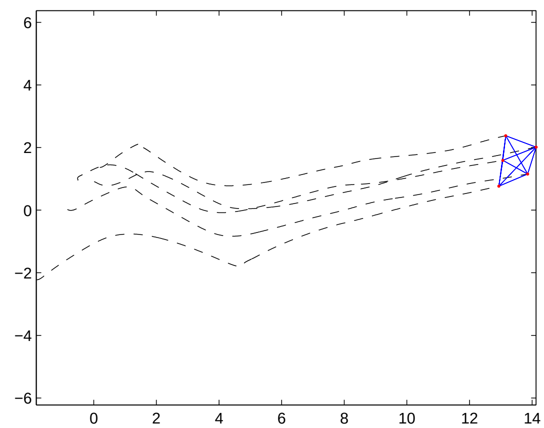

Fig. 7. Sim. time $t = 12$ sec

trolled lagrangian and controlled hamiltonian systems. *ESAIM: Control, Optimization and Calculus of Variations*, 8:393–422, 2002.

[3] Jorge Cortes, Sonia Martinez, Timur Karatas, and Francesco Bullo. Coverage control for mobile sensing networks. *IEEE Transactions on Robotics and Automation*, 2002. (submitted).

[4] J. P. Desai, J. P. Ostrowski, and V. Kumar. Modeling and control of formations of nonholonomic mobile robots. *IEEE Transactions on Robotics and Automation*, 17(6):905–908, 2001.

[5] J. Alexander Fax and Richard M. Murray. Graph Laplacians and stabilization of vehicle formations. *15th IFAC Congress, Barcelona, Spain*, 2002.

[6] G. Flierl, D. Grunbaum, S. Levin, and D. Olson. From individuals to aggregations: the interplay between behavior and physics. *Journal of Theoretical Biology*, 196:397–454, 1999.

[7] V. Gazi and K. M. Passino. Stability analysis of swarms. *IEEE Transactions on Automatic Control*, 48(4):692–696, April 2003.

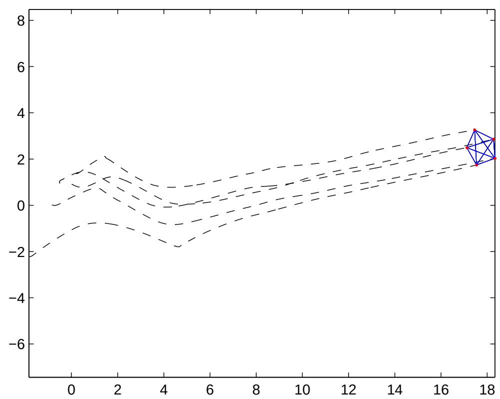

Fig. 8. Sim. time $t = 15$ sec

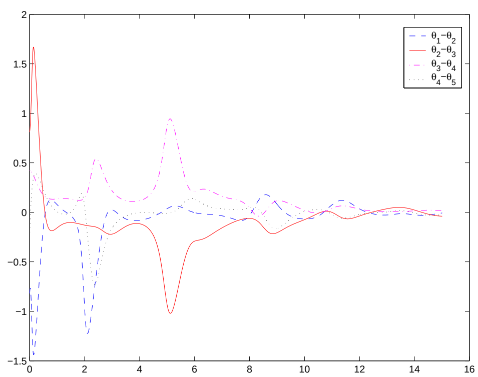

Fig. 9. Relative heading trajectories.

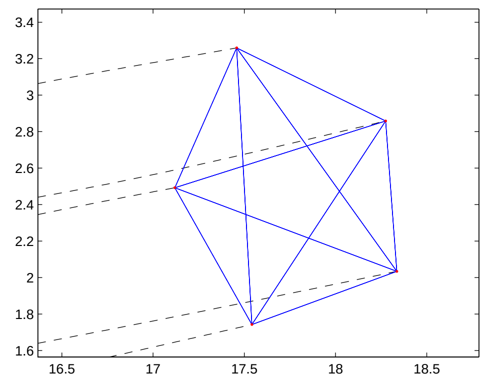

Fig. 10. Convergence of inter-agent distances.

[8] C. Godsil and G. Royle. *Algebraic Graph Theory*. Springer Graduate Texts in Mathematics # 207, New York, 2001.

[9] A. Jadbabaie, J. Lin, and A. S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *Accepted for publication in IEEE Transactions on Automatic Control*, July 2002.

[10] N. Leonard and E. Friorelli. Virtual leaders, artificial potentials and coordinated control of groups. *IEEE Conference on Decision and Control*, 2001.

[11] H. Levine and W. J. Rappel. Self organization in systems of self-propelled particles. *Physical Review E*, 63:208–211, 2001.

[12] Y. Liu, K. M. Passino, and M. M. Polycarpou. Stability analysis of m-dimensional asynchronous swarms with a fixed communication topology. *IEEE Transactions on Automatic Control*, 48(1):76–95, February 2003.

[13] A. S. Mikhailov and D. Zannette. Noise induced breakdown of collective coherent motion in swarms. *Physical Review E*, 60:4571–4575, 1999.

[14] P. Ögren, M. Egerstedt, and X. Hu. A control Lyapunov function approach to multi-agent coordination. *IEEE Transactions on Robotics and Automation*, 18, October 2002.

[15] R. Olfati and R. M. Murray. Distributed structural stabilization and tracking for formations of dynamic multi-agents. In *IEEE Conference on Decision and Control*, Las Vegas, NV, 2002.

[16] C. Reynolds. Flocks, birds, and schools: a distributed behavioral model. *Computer Graphics*, 21:25–34, 1987.

[17] N. Shimoyama, K. Sugawa, T. Mizuguchi, Y. Hayakawa, and M. Sano. Collective motion in a system of motile elements. *Physical Review Letters*, 76:3879–3873, 1996.

[18] P. Tabuada, G. J. Pappas, and P. Lima. Feasible formations of multi-agent systems. In *Proceedings of the American Control Conference*, Arlington, VA, 2001.

[19] Herbert G. Tanner, Ali Jadbabaie, and George J. Pappas. Stable flocking of mobile agents, Part I: Fixed topology. In *Proceedings of the IEEE Conference on Decision and Control*, 2003. (submitted).

[20] Herbert G. Tanner, Ali Jadbabaie, and George J. Pappas. Stable flocking of mobile agents, Part II: Dynamic topology. In *Proceedings of the IEEE Conference on Decision and Control*, 2003. (submitted).

[21] J. Toner and Y. Tu. Long range order in a two dimensional xy model: How birds fly together. *Physical Review Letters*, 75:4326–4329, 1995.

[22] T. Vicsek, A. Czirok, E. Ben Jacob, I. Cohen, and O. Schochet. Novel type of phase transitions in a system of self-driven particles. *Physical Review Letters*, 75:1226–1229, 1995.

[23] K. Warburton and J. Lazarus. Tendency-distance models of social cohesion in animal groups. *Journal of Theoretical Biology*, 150:473–488, 1991.