Approximate equivalence and approximate synchronization of metric transition systems

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***Abstract*— In this paper, we consider metric transition systems which are transition systems equipped with metrics for observation and synchronization labels. The existence of metrics leads to the introduction of two new concepts, (i) $(\epsilon, \delta)-$ approximate (bi)simulation of transition systems and (ii) approximate synchronization of transition systems.**

We show that the notion of $(\epsilon, \delta)-$ approximate (bi)simulation can be thought of as a generalization or relaxation of the earlier work on $\delta-$ approximate (bi)simulation by Girard and Pappas. We demonstrate the link between reachability verification and approximate (bi)simulation, and we also provide a characterization of (bi)simulation relations using a tool similar to the (bi)simulation function.

Approximate synchronization can be thought of as a generalization of synchronization of transition systems in the usual sense. In fact, the usual synchronization and interleaving synchronization are two special cases of the notion of approximate synchronization developed in this paper. Furthermore, we present a result on the compositional properties of the approximate (bi)simulation with respect to the approximate synchronization.

I. INTRODUCTION

System abstraction is an important tool for analyzing complex systems. With abstraction, the complexity of the systems (typically associated with the size of the state space) can be decreased, resulting in lesser computational cost in the analysis [1], [2], [3].

System abstraction is traditionally associated with system equivalence, in the sense abstraction of a complex system amounts to constructing an equivalent system with lesser complexity. The equivalence guarantees that the results of analysis performed on the less complex system can be carried over into the complex system. Language equivalence and bisimulation (and its variants) are two of the most commonly used notion of system equivalence for systems abstraction [4], [5], [6], [7].

Requiring the abstraction to be equivalent to the original system is sometimes too restrictive. Researchers have been working to develop more relaxed abstraction theories that enable further model simplification. One of the ideas is to relax the requirement that the abstraction is equivalent to the original system, and replace with that the abstraction is only *approximately* equal to the original system (see, e.g. [8], [9], [10], [11]). The key ingredient to these theories is a metric that can quantify the distance between the system and its abstraction, and hence the quality of the abstraction. In this paper, we start with the idea of approximate bisimulation of transition systems as developed recently in [11], [12], [13]. Transition systems is a convenient framework to use because many interesting classes of dynamical systems can be embedded as transition systems, and abstraction can be studied as abstraction of the transition system [14], [5].

In this paper, we extend the previous work by introducing a pseudo-metric on set of labels of the transition systems. Having a notion of distance in the set of labels enables us to define a notion of *approximate synchronization*. Loosely speaking, by approximate synchronization we mean allowing systems to synchronize not only on the same label, but also with labels that are close. Approximate synchronization can be thought of as a relaxation of the notion of synchronization in the usual sense.

Contrary to *exact* notions of synchronization for traditional transition systems, *approximate* synchronization is a much more natural and robust concept especially when different system need to synchronize over temporal or spatial variables where exact synchronization may be too restrictive or not robust. For example, random communication delays between geographically distant subsystems requires a notion of synchronization that does not require strict simultaneity. Thus, approximation in the synchronization can be related to tolerance in timing. Similarly, in the area of multi-agent control, if spatial information about the agents is captured on the labels, then approximate synchronization can be used as a compact and natural way of representing communication (or cooperation) range.

In this paper, we first extend the notion of approximate (bi)simulation of metric transition systems, by introducing a pseudometric on the set of labels. We elucidate the relation between our work and an earlier work by Girard and Pappas [11], [12], and we also provide a way to characterize approximate (bi)simulation relations by using an extension of the (bi)simulation functions. We then introduce the notion of approximate synchronization and present a result that shows that approximate (bi)simulation is compositional with respect to approximate synchronization. Even further, we show that this result also extends to the case where clusters of transition systems (called composite transition systems) are synchronized.

The remaining of the paper is organized as follows. Section II presents the extension of approximate (bi)simulation by including a pseudometric in the set of labels of the transition systems. In Section III we present a way to characterize the approximate (bi)simulation relations discussed in the preceding section, by means of bisimulation functions.

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Section IV is devoted for introducing the idea of approximate synchronization and presenting its properties. Here we also present the compositional properties of approximate (bi)simulation under approximate synchronization. In Section V we give some concluding remarks and possible future research directions.

II. METRIC TRANSITION SYSTEMS

In this section, we extend the idea of approximate simulation and bisimulation, by introducing a pseudometric on the set of labels of the transition systems.

We define a transition system as a six tuple $T = (Q, \Sigma, \rightarrow, Q^0, \Pi, \langle\cdot\rangle)$, where Q is the set of states, Σ is the set of labels, $\rightarrow \subset Q \times \Sigma \times Q$ is a set of transitions, Q^0 is the set of initial states, Π is the set of possible observations, $\langle\cdot\rangle : Q \rightarrow \Pi$ is the observation map. The transition system is called a *metric transition system* if the set of observations Π and labels Σ are equipped with pseudometrics d_Π and d_Σ respectively[1].

Notation 2.1: In this paper we shall use the following notations.

$$\forall\varepsilon \geq 0, \sigma \in \Sigma,\ B_\varepsilon(\sigma) := \{\sigma' \in \Sigma \mid d_\Sigma(\sigma, \sigma') \leq \varepsilon\},$$
$$\forall\varepsilon \geq 0, z \in \Pi,\ B_\varepsilon(z) := \{z' \in \Pi \mid d_\Pi(z, z') \leq \varepsilon\},$$
$$\forall q \in Q, S \subset \Sigma,\ \Omega(q, S) := \{q' \in Q \mid \exists\sigma \in S, q \xrightarrow{\sigma} q'\}.$$

Definition 2.2: Given two transition systems $T_i = (Q_i, \Sigma, \rightarrow_i, Q_i^0, \Pi, \langle\cdot\rangle_i)$, $i = 1, 2$. A relation $\mathcal{R} \subset Q_1 \times Q_2$ is a $(\varepsilon, \delta)-$ **approximate simulation** of T_1 by T_2, $\delta, \varepsilon \geq 0$, if for any $(q_1, q_2) \in \mathcal{R}$,
(i) $d_\Pi(\langle q_1\rangle_1, \langle q_2\rangle_2) \leq \delta$,
(ii) For any $a \in \Sigma, q_1' \in Q_1$ such that $q_1 \xrightarrow{a} q_1'$, there exists an $a' \in \Sigma$ and $q_2' \in Q_2$ such that

$$d_\Sigma(a, a') \leq \varepsilon,\ q_2 \xrightarrow{a'} q_2',\ (q_1', q_2') \in \mathcal{R}.$$

Notice that ε and δ represent the precision in the approximation in terms of the synchronization labels and the observations respectively. A $(0, \delta)-$approximate simulation relation is a $\delta-$ approximate simulation in the sense of [11], which requires exact synchronization. A $(0, 0)-$ approximate simulation relation is a classical exact simulation relation with exact synchronization. Furthermore, the following proposition reveals the partial ordering of approximate simulation relations.

Proposition 2.3: Given two transition systems $T_i = (Q_i, \Sigma, \rightarrow_i, Q_i^0, \Pi, \langle\cdot\rangle_i)$, $i = 1, 2$. Let $\mathcal{R} \subset Q_1 \times Q_2$. For any $\delta' \geq \delta \geq 0$ and $\varepsilon' \geq \varepsilon \geq 0$ the following statements hold.
(i) If $\mathcal{R}$ is a $(\varepsilon, \delta)-$approximate simulation of T_1 by T_2 then it is also a $(\varepsilon', \delta)-$ approximate simulation of T_1 by T_2.
(ii) If $\mathcal{R}$ is a $(\varepsilon, \delta)-$approximate simulation of T_1 by T_2 then it is also a $(\varepsilon, \delta')-$ approximate simulation of T_1 by T_2.

A $(\varepsilon, \delta)-$approximate bisimulation relation can be defined as a symmetric version of a $(\varepsilon, \delta)-$approximate simulation, as follows.

[1]From this point on we assume that all transition systems are metric transition systems, hence we do not distinguish between the two notions

Definition 2.4: Given two transition systems $T_i = (Q_i, \Sigma, \rightarrow_i, Q_i^0, \Pi, \langle\cdot\rangle_i)$, $i = 1, 2$. A relation $\mathcal{R} \subset Q_1 \times Q_2$ is a $(\varepsilon, \delta)-$ **approximate bisimulation** between T_1 and T_2, $\delta, \varepsilon \geq 0$, if $\mathcal{R}$ is both a $(\varepsilon, \delta)-$ **approximate simulation** of T_1 by T_2 and a $(\varepsilon, \delta)-$ **approximate simulation** of T_2 by T_1.

Corollary 2.5: Given two transition systems $T_i = (Q_i, \Sigma, \rightarrow_i, Q_i^0, \Pi, \langle\cdot\rangle_i)$, $i = 1, 2$. Let $\mathcal{R} \subset Q_1 \times Q_2$. For any $\delta' \geq \delta \geq 0$ and $\varepsilon' \geq \varepsilon \geq 0$ the following statements hold.
(i) If $\mathcal{R}$ is a $(\varepsilon, \delta)-$ approximate bisimulation between T_1 and T_2 then it is also a $(\varepsilon', \delta)-$ approximate bisimulation between T_1 and T_2.
(ii) If $\mathcal{R}$ is a $(\varepsilon, \delta)-$ approximate bisimulation between T_1 and T_2 then it is also a $(\varepsilon, \delta')-$ approximate bisimulation between T_1 and T_2.

Approximate simulation and bisimilarity between transition systems are characterized as follows.

Definition 2.6: Given two transition systems $T_i = (Q_i, \Sigma, \rightarrow_i, Q_i^0, \Pi, \langle\cdot\rangle_i)$, $i = 1, 2$. We say that T_2 **simulates** T_1 **with precision** (ε, δ) if there exists $\mathcal{R}$, a $(\varepsilon, \delta)-$ approximate simulation of T_2 by T_1, such that for every $q_1^0 \in Q_1^0$, there exists a $q_2^0 \in Q_2^0$ such that $(q_1^0, q_2^0) \in \mathcal{R}$. This relation is denoted by $T_1 \preceq_{\varepsilon,\delta} T_2$.

Definition 2.7: Given two transition systems $T_i = (Q_i, \Sigma, \rightarrow_i, Q_i^0, \Pi, \langle\cdot\rangle_i)$, $i = 1, 2$. We say that T_1 and T_2 **are approximately bisimilar with precision** (ε, δ) if there exists $\mathcal{R}$, a $(\varepsilon, \delta)-$ approximate bisimulation between T_1 and T_2, such that
(i) for every $q_1^0 \in Q_1^0$, there exists a $q_2^0 \in Q_2^0$ such that $(q_1^0, q_2^0) \in \mathcal{R}$,
(ii) for every $q_2^0 \in Q_2^0$, there exists a $q_1^0 \in Q_1^0$ such that $(q_1^0, q_2^0) \in \mathcal{R}$.
This relation is denoted by $T_1 \approx_{\varepsilon,\delta} T_2$.

The concept of $(\varepsilon, \delta)-$ approximate bisimulation is illustrated in Figure 1. Based on Proposition 2.3 and Corollary 2.5, we can derive the following proposition.

Proposition 2.8: Given two transition systems T_1 and T_2. For any $\delta' \geq \delta \geq 0$ and $\varepsilon' \geq \varepsilon \geq 0$. the following statements hold.
(i) If $T_1 \preceq_{\varepsilon,\delta} T_2$ then $T_1 \preceq_{\varepsilon',\delta} T_2$.
(ii) If $T_1 \preceq_{\varepsilon,\delta} T_2$ then $T_1 \preceq_{\varepsilon,\delta'} T_2$.
(iii) If $T_1 \approx_{\varepsilon,\delta} T_2$ then $T \approx_{\varepsilon',\delta} T_2$.
(iv) If $T_1 \approx_{\varepsilon,\delta} T_2$ then $T \approx_{\varepsilon,\delta'} T_2$.

For any $\varepsilon, \delta \geq 0$, the approximate bisimulation relation $\approx_{\varepsilon,\delta}$ is clearly reflexive and symmetric, i.e. for any transition systems T_1 and T_2,
(reflexive) $T_1 \approx_{\varepsilon,\delta} T_1$.
(symmetric) If $T_1 \approx_{\varepsilon,\delta} T_2$, then $T_2 \approx_{\varepsilon,\delta} T_1$.
Another property of interest is the transitivity property of the approximate simulation and bisimulation.

Proposition 2.9: Given three transition systems T_1, T_2 and T_3. For any $\delta, \delta' \geq 0$ and $\varepsilon, \varepsilon' \geq 0$. the following statements hold.
(i) If $T_1 \preceq_{\varepsilon,\delta} T_2$ and $T_2 \preceq_{\varepsilon',\delta'} T_3$, then $T_1 \preceq_{\varepsilon+\varepsilon',\delta+\delta'} T_3$.
(ii) If $T_1 \approx_{\varepsilon,\delta} T_2$ and $T_2 \approx_{\varepsilon',\delta'} T_3$, then $T_1 \approx_{\varepsilon+\varepsilon',\delta+\delta'} T_3$.

Proof: We only prove (i), since (ii) can be proven

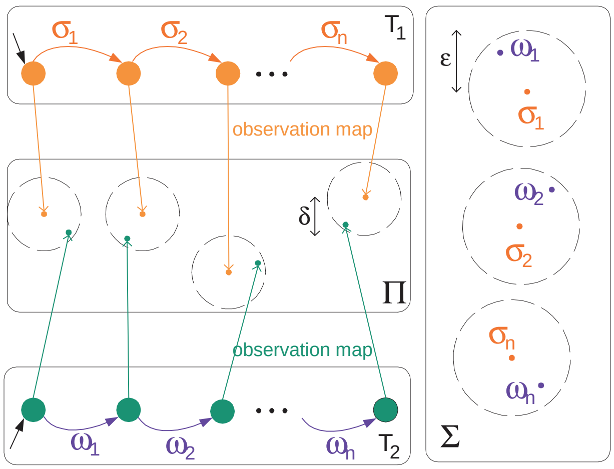

Fig. 1. An illustration of approximate (bi)simulation with labels between two transition systems T_1 and T_2. The outputs of related states must be within at most δ. The two transition systems do not have to synchronize with the same labels. Rather, the labels can be at most ε apart.

in an analogous manner. Suppose that $T_1 \preceq_{\varepsilon,\delta} T_2$ and $T_2 \preceq_{\varepsilon',\delta'} T_3$, and $\mathcal{R}_{12}$ and $\mathcal{R}_{23}$ are the $(\varepsilon,\delta)-$ **approximate simulation** of T_1 by T_2, and $(\varepsilon',\delta')-$ **approximate simulation** of T_2 by T_3 respectively. We shall prove that

$$\begin{aligned}\mathcal{R}_{13} &:= \mathcal{R}_{12} \circ \mathcal{R}_{23}, \\ &= \{(q_1,q_3) \mid \exists q_2, (q_1,q_2) \in \mathcal{R}_{12}, (q_2,q_3) \in \mathcal{R}_{23}\} \quad (1)\end{aligned}$$

is a $(\varepsilon+\varepsilon',\delta+\delta')-$ **approximate simulation** of T_2 by T_3. Take any $(q_1,q_3) \in \mathcal{R}_{23}$. First, we show that

$$d_\Pi(\langle q_1\rangle_1, \langle q_3\rangle_3) \le \delta + \delta'. \quad (2)$$

By definition of $\mathcal{R}_{13}$, there exists a $q_2 \in Q_2$ such that $(q_1,q_2) \in \mathcal{R}_{12}$ and $(q_2,q_3) \in \mathcal{R}_{23}$. From there we can infer that

$$\begin{aligned}d_\Pi(\langle q_1\rangle_1, \langle q_2\rangle_2) &\le \delta, \\ d_\Pi(\langle q_2\rangle_2, \langle q_3\rangle_3) &\le \delta'.\end{aligned}$$

Equation (2) follows because of the pseudometric properties. Now we shall show that if $q_1 \xrightarrow{\sigma} q_1'$ for some $\sigma \in \Sigma$ and $q_1' \in Q_1$, then there exist $\sigma' \in \Sigma$ and $q_3' \in Q_3$ such that

$$(q_1',q_3') \in \mathcal{R}_{23}, q_3 \xrightarrow{\sigma'} q_3', d_\Sigma(\sigma,\sigma') \le \varepsilon+\varepsilon'. \quad (3)$$

By the existence of a $q_2 \in Q_2$ as above, we can infer the existence of a $q_2' \in Q_2$ and $\sigma'' \in \Sigma$ such that

$$(q_1',q_2') \in \mathcal{R}_{12}, q_2 \xrightarrow{\sigma''} q_2', d_\Sigma(\sigma,\sigma'') \le \varepsilon.$$

However, this in turn implies the existence of a $q_3' \in Q_3$ and $\sigma' \in \Sigma$ such that

$$(q_2',q_3') \in \mathcal{R}_{23}, q_3 \xrightarrow{\sigma'} q_3', d_\Sigma(\sigma',\sigma'') \le \varepsilon'.$$

Again, (3) follows immediately from the definition of $\mathcal{R}_{23}$ and the pseudometric properties. ∎

The relation between the reachable sets (of observations) of the transition systems and the approximate (bi)simulation is summarized as follows.

Definition 2.10: Given a transition system $T = (Q, \Sigma, \rightarrow, Q^0, \Pi, \langle\cdot\rangle)$, an observation $y \in \Pi$ belongs to the reachable set of the transition system $\mathcal{R}(T)$ if there exists an initial state $x_0 \in Q^0$ and a trajectory starting from x_0,

$$x_0 \xrightarrow{a_1} x_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} x_n,$$

such that $\langle x_n\rangle = y$.

Theorem 2.11: Given two transition systems T_1 and T_2, the following relations hold.

(i) $T_1 \preceq_{\varepsilon,\delta} T_2$ for some $\varepsilon,\delta \ge 0$ implies

$$\sup_{y_1\in\mathcal{R}(T_1)} \inf_{y_2\in\mathcal{R}(T_2)} d_\Pi(y_1,y_2) \le \delta. \quad (4)$$

(ii) $T_1 \approx_{\varepsilon,\delta} T_2$ for some $\varepsilon,\delta \ge 0$ implies

$$\max\left(\sup_{y_1\in\mathcal{R}(T_1)} \inf_{y_2\in\mathcal{R}(T_2)} d_\Pi(y_1,y_2), \sup_{y_2\in\mathcal{R}(T_2)} \inf_{y_1\in\mathcal{R}(T_1)} d_\Pi(y_1,y_2)\right) \le \delta. \quad (5)$$

Proof: (i) We need to show that if $T_1 \preceq_{\varepsilon,\delta} T_2$ for some $\varepsilon,\delta \ge 0$, then for any $y_1 \in \mathcal{R}(T_1)$, there exists a $y_2 \in \mathcal{R}(T_2)$ such that $d_\Pi(y_1,y_2) \le \delta$. There exists a trajectory of T_1 starting from $x_{1,0} \in Q_1^0$,

$$x_{1,0} \xrightarrow{a_1} x_{1,1} \xrightarrow{a_2} \cdots \xrightarrow{a_n} x_{1,n},$$

such that $\langle x_{1,n}\rangle_1 = y_1$. Suppose that $\mathcal{R} \subset Q_1 \times Q_2$ is a (ε,δ) -approximate simulation of T_1 by T_2. By the definition of approximate simulation, we can infer that there exists a trajectory of T_2 starting from a $x_{1,0} \in Q_1^0$,

$$\begin{gathered}x_{2,0} \xrightarrow{a_1'} x_{2,1} \xrightarrow{a_2'} \cdots \xrightarrow{a_n'} x_{2,n}, \\ (x_{1,i}, x_{2,i}) \in \mathcal{R}.\end{gathered}$$

Denote $\langle x_{2,n}\rangle_2 = y_2$. It follows from the definition of approximate simulation that $d_\Pi(y_1,y_2) \le \delta$.

(ii) Analogous to part (i). ∎

The application of approximate (bi)simulation as an aid in safety verification of dynamical systems is presented in [11], [12]. Given a dynamical system embedded as a transition system T_1, another dynamical system embedded as a transition system T_2 is constructed such that $T_1 \preceq_{0,\delta} T_2$. The system corresponding with T_2 is simpler, in the sense of smaller state space. The reachable set of T_1 can thus be approximated with that of T_2 with precision δ.

The introduction of a metric for the labels can be thought of as a relaxation that allows for tighter bound in the approximation of the reachable set. This is illustrated on the continuous time dynamical system

$$\frac{dx}{dt} = f(x,u), y = h(x), \quad (6)$$

$$x \in \mathcal{X}, x(0) \in \mathcal{X}^0, u \in \mathcal{U}, y \in \mathcal{Y} \subset \mathbb{R}^m. \quad (7)$$

This system can be embedded into a transition system $T = (Q, \Sigma, \rightarrow, Q^0, \Pi, \langle\cdot\rangle)$, where $Q = \mathcal{X}$, $\Sigma = \mathbb{R}_+$, $Q^0 = \mathcal{X}^0$, $\Pi = \mathcal{Y}$, $\langle x \rangle = h(x)$.

$$\rightarrow \subset \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}^n,$$

such that $x \xrightarrow{\tau} x'$ if and only if there exist $x_0 \in \mathcal{X}^0$ and $u : [0, \tau] \rightarrow \mathcal{U}$ such that the continuous solution to the differential equation

$$\frac{dx}{dt} = f(x, u), x(0) = x_0 \tag{8}$$

satisfies $x(\tau) = x'$. Alternatively stated, $x \xrightarrow{\tau} x'$ if and only if there is an input that can drive the system starting at the initial state x to the state x' in τ time unit. The set of labels and observations, $\mathbb{R}_+$ and $\mathcal{Y} \subset \mathbb{R}^m$ are equipped with the Euclidian distance $\|\cdot\|$. With this interpretation of transition system, the following implication can be proven.

Proposition 2.12: Given two transition systems T_1 and T_2, the following relations hold.
(i) $T_1 \preceq_{\infty,\delta} T_2$ for some $\delta \geq 0$ if and only if

$$\sup_{y_1 \in \mathcal{R}(T_1)} \inf_{y_2 \in \mathcal{R}(T_2)} d_\Pi(y_1, y_2) \leq \delta. \tag{9}$$

(ii) $T_1 \approx_{\infty,\delta} T_2$ for some $\delta \geq 0$ if and only if

$$\max\left(\sup_{y_1 \in \mathcal{R}(T_1)} \inf_{y_2 \in \mathcal{R}(T_2)} d_\Pi(y_1, y_2), \sup_{y_2 \in \mathcal{R}(T_2)} \inf_{y_1 \in \mathcal{R}(T_1)} d_\Pi(y_1, y_2) \right) \leq \delta. \tag{10}$$

Therefore, by relaxing the requirement on the labels, we can get a result stronger than Theorem 2.11. A different treatment of a similar idea is presented in [15].

III. Extension of the (bi)simulation functions

In this section we discuss the extension of the concept of (bi)simulation functions [11], to deal with metrics on synchronization labels.

Definition 3.1: Given two transition systems $T_i = (Q_i, \Sigma, \rightarrow_i, Q_i^0, \Pi, \langle\cdot\rangle_i)$, $i = 1, 2$. A function $\phi : Q_1 \times Q_2 \rightarrow \mathbb{R}_+ \cup \{\infty\}$ is an ε - simulation function of T_1 by T_2 if for any $q_1 \in Q_1$ and $q_2 \in Q_2$,

$$\phi(q_1, q_2) \geq d_\Pi(\langle q_1 \rangle_1, \langle q_2 \rangle_2), \tag{11a}$$

$$\phi(q_1, q_2) \geq \sup_{q_1 \xrightarrow{\sigma} q_1'} \inf_{q_2 \xrightarrow{B_\varepsilon(\sigma)} q_2'} \phi(q_1', q_2'). \tag{11b}$$

Notice that an $\varepsilon-$simulation function can be thought of as a relaxed version of bisimulation function in the sense of [11]. In order the match a transition of T_1, T_2 does not necessarily perform a transition with the same label. Rather, T_2 can choose any move, as long as its label is at most ε apart from that of T_1. A bisimulation function in the sense of [11] is a $0-$ simulation function.

Proposition 3.2: Given two transition systems T_1 and T_2. If ϕ is an $\varepsilon-$simulation function of T_1 by T_2, for some $\varepsilon \geq 0$, then it is also an $\varepsilon'-$simulation function of T_1 by T_2, for any $\varepsilon' \geq \varepsilon \geq 0$.

Definition 3.3: Given two transition systems $T_i = (Q_i, \Sigma, \rightarrow_i, Q_i^0, \Pi, \langle\cdot\rangle_i)$, $i = 1, 2$. A function $\phi : Q_1 \times Q_2 \rightarrow \mathbb{R}_+ \cup \{\infty\}$ is an $\varepsilon-$ bisimulation function between T_1 and T_2 if it is both an ε - simulation function of T_1 by T_2 and an ε - simulation function of T_2 by T_1. That is, for any $q_1 \in Q_1$ and $q_2 \in Q_2$,

$$\phi(q_1, q_2) \geq d_\Pi(\langle q_1 \rangle_1, \langle q_2 \rangle_2), \tag{12}$$

$$\phi(q_1, q_2) \geq \sup_{q_1 \xrightarrow{\sigma} q_1'} \inf_{q_2 \xrightarrow{B_\varepsilon(\sigma)} q_2'} \phi(q_1', q_2'), \tag{13}$$

$$\phi(q_1, q_2) \geq \sup_{q_2 \xrightarrow{\sigma} q_2'} \inf_{q_1 \xrightarrow{B_\varepsilon(\sigma)} q_1'} \phi(q_1', q_2'). \tag{14}$$

The relation between (bi)simulation functions and approximate (bi)simulation can be summarized in the following theorems.

Theorem 3.4: Given two transition systems T_1 and T_2. If ϕ is an $\varepsilon-$ simulation function of T_1 by T_2, for some $\varepsilon \geq 0$, then for any $\delta \geq 0$, its $\delta-$ level set,

$$\mathcal{R}_\delta(\phi) := \{(q_1, q_2) \mid \phi(q_1, q_2) \leq \delta\},$$

is a $(\varepsilon, \delta)-$ **approximate simulation** of T_1 by T_2.

Proof: Take any $(q_1, q_2) \in \mathcal{R}_\delta(\phi)$, by (11a) we have that,

$$d_\Pi(\langle q_1 \rangle_1, \langle q_2 \rangle_2) \leq \delta. \tag{15}$$

For any $\sigma \in \Sigma$ such that $q_1 \xrightarrow{\sigma} q_1'$, (11b) implies the existence of $q_2' \in Q_2$ and $\sigma' \in \Sigma$ such that

$$q_2 \xrightarrow{\sigma'} q_2', d_\Sigma(\sigma, \sigma') \leq \varepsilon,$$
$$\phi(q_1', q_2') \leq \delta.$$

Therefore $(q_1', q_2') \in \mathcal{R}_\delta(\phi)$. ■

Theorem 3.5: Given two transition systems T_1 and T_2. If ϕ is an $\varepsilon-$ bisimulation function between T_1 and T_2, for some $\varepsilon \geq 0$, then for any $\delta \geq 0$, its $\delta-$ level set,

$$\mathcal{R}_\delta(\phi) := \{(q_1, q_2) \mid \phi(q_1, q_2) \leq \delta\},$$

is a $(\varepsilon, \delta)-$ **approximate bisimulation** between T_1 and T_2.

Proof: Analogous to that of Theorem 3.4. ■

Generally speaking, the characterization of an $\varepsilon-$ simulation function is similar to that of a simulation function when there is nondeterminism in the system.

IV. Approximate synchronization

Typically, synchronization of transition systems is formalized by (exact) synchronization of the labels. In this section, we introduce the idea of approximate synchronization. Loosely speaking, the idea is to let two transition systems synchronize using labels that are close, but not necessarily equal. Closeness is defined in the sense of the a pseudometric in the set of labels.

A. Approximate synchronization of transition systems

Definition 4.1: Given two transition systems $T_i = (Q_i, \Sigma, \rightarrow_i, Q_i^0, \Pi_i, \langle\cdot\rangle_i)$, $i = 1, 2$. The **approximate synchronization** operator $\|_\varepsilon$, $\varepsilon \geq 0$, acting on the two systems results in another transition system

$$T := T_1 \parallel_\varepsilon T_2, \tag{16}$$

where $T = (Q_1 \times Q_2, \Sigma \times \Sigma, \rightarrow, Q_1^0 \times Q_2^0, \Pi_1 \times \Pi_2, \langle\cdot\rangle)$. The transition relation $\rightarrow$ is such that $(q_1, q_2) \overset{\sigma,\sigma'}{\rightarrow} (q_1', q_2')$ iff $q_1 \overset{\sigma}{\rightarrow}_1 q_1'$, $q_2 \overset{\sigma'}{\rightarrow}_2 q_2'$, $d_\Sigma(\sigma, \sigma') \leq \varepsilon$. The observation map $\langle\cdot\rangle$ is defined as

$$\langle (q_1, q_2) \rangle := (\langle q_1 \rangle_1, \langle q_2 \rangle_2). \tag{17}$$

Notice that the composite transition system $T = T_1 \parallel_\varepsilon T_2$ is quite different from the transition systems T_1 and T_2, in the following sense:

- The observation space of T is a product of those of T_1 and T_2.
- The set of labels of T is also a product of those of T_1 and T_2.

We need to define a notion of pseudometric for an observation space that is a product of two observation spaces, and similarly for the set of labels.

Definition 4.2: The observation space $\Pi_1 \times \Pi_2$ is equipped with the following pseudometric.

$$d_\Pi((\pi_1, \pi_2), (\pi_1', \pi_2')) := d_{\Pi_1}(\pi_1, \pi_1') + d_{\Pi_2}(\pi_2, \pi_2'). \tag{18}$$

The set of labels $\Sigma \times \Sigma$ is equipped with the following pseudometric.

$$d_{\Sigma^2}((\sigma_1, \sigma_2), (\sigma_1', \sigma_2')) := \max_{i=1,2} \max_{j=1,2} d_\Sigma(\sigma_i, \sigma_j'). \tag{19}$$

Approximate synchronization can be thought of as a relaxed version of the exact synchronization. Exact synchronization is a special case of approximate synchronization $\parallel_\varepsilon$, namely when $\varepsilon = 0$. Obviously, the larger the tolerance (ε) in the synchronization is, the more flexible the two systems can evolve with respect to each other. If we assume that the transition systems have stutter transition [16], the case when $\varepsilon = \infty$ can be thought of as the situation when the executions of the two transition systems are interleaving. The executions can interleave because one transition system can always synchronize with the stutter transition of the other.

The fact that defined notion of approximate synchronization is a relaxation of the traditional notion of synchronization is reflected in the following proposition.

Proposition 4.3: Given two transition systems $T_i = (Q_i, \Sigma, \rightarrow_i, Q_i^0, \Pi, \langle\cdot\rangle_i)$, $i = 1, 2$. For any $\varepsilon, \varepsilon' \geq 0$, the following holds.

$$T_1 \parallel_\varepsilon T_2 \preceq_{0,0} T_1 \parallel_{\varepsilon+\varepsilon'} T_2. \tag{20}$$

Proof: The identity relation in $(Q_1 \times Q_2) \times (Q_1 \times Q_2)$ is a $(0, 0)-$ approximate simulation relation of T by T'. ■

This proposition tells us that a synchronization with higher tolerance always simulates one with less tolerance.

It is already known that the notion of approximate (bi)simulation has a compositional property [11] with respect to exact synchronization. In the following we shall show that the extended notion of approximate (bi)simulation that we present in this paper also has a compositional property with respect to approximate synchronization.

Theorem 4.4: Consider transition systems T_1, T_2, T_1' and T_2'. Suppose that the transition systems T_1 and T_1' have observation space Π_1, while T_2 and T_2' have observation space Π_2. Moreover we assume that all of them share the same set of labels Σ. If $T_1 \preceq_{\varepsilon_1,\delta_1} T_1'$ and $T_2 \preceq_{\varepsilon_2,\delta_2} T_2'$, then for any $\varepsilon \geq 0$,

$$T_1 \parallel_\varepsilon T_2 \preceq_{\varepsilon+\max(\varepsilon_1,\varepsilon_2),\delta_1+\delta_2} T_1' \parallel_{\varepsilon+\varepsilon_1+\varepsilon_2} T_2'. \tag{21}$$

Proof: Denote

$$T := T_1 \parallel_\varepsilon T_2, T' := T_1' \parallel_{\varepsilon+\varepsilon_1+\varepsilon_2} T_2'. \tag{22}$$

Since $T_1 \preceq_{\varepsilon_1,\delta_1} T_1'$ and $T_2 \preceq_{\varepsilon_2,\delta_2} T_2'$, there exist appropriate approximate simulation relations $\mathcal{R}_1 \subset Q_1 \times Q_1'$ and $\mathcal{R}_2 \subset Q_2 \times Q_2'$ (see Definition 2.6). We define $\mathcal{R} \subset (Q_1 \times Q_2) \times (Q_1' \times Q_2')$ as follows.

$$((q_1, q_2), (q_1', q_2')) \in \mathcal{R} :\Leftrightarrow$$
$$(q_1, q_1') \in \mathcal{R}_1 \text{ and } (q_2, q_2') \in \mathcal{R}_2.$$

We are going to prove that $\mathcal{R}$ is a $(\varepsilon + \max(\varepsilon_1, \varepsilon_2), \delta_1 + \delta_2)-$ approximate simulation of T by T'. Take any $((q_1, q_2), (q_1', q_2')) \in \mathcal{R}$.

$$\begin{aligned} & d_\Pi((\langle q_1 \rangle_1, \langle q_2 \rangle_2), (\langle q_1' \rangle_{1'}, \langle q_2' \rangle_{2'})) = \\ & = d_{\Pi_1}(\langle q_1 \rangle_1, \langle q_1' \rangle_{1'}) + d_{\Pi_2}(\langle q_1 \rangle_2, \langle q_1' \rangle_{2'}) \\ & \leq \delta_1 + \delta_2. \end{aligned} \tag{23}$$

The inequality is due to the fact that $(q_i, q_i') \in \mathcal{R}_i$, $i = 1, 2$.

For any $\alpha, \beta \in \Sigma$ and $(\tilde{q}_1, \tilde{q}_2) \in Q_1 \times Q_2$ such that

$$d_\Sigma(\alpha, \beta) \leq \varepsilon, (q_1, q_2) \overset{\alpha,\beta}{\rightarrow}_T (\tilde{q}_1, \tilde{q}_2),$$

we need to show that there exist $\alpha', \beta' \in \Sigma$ and $(\tilde{q}_1', \tilde{q}_2') \in Q_1' \times Q_2'$ such that

$$d_\Sigma(\alpha', \beta') \leq \varepsilon + \varepsilon_1 + \varepsilon_2, (q_1', q_2') \overset{\alpha',\beta'}{\rightarrow}_{T'} (\tilde{q}_1', \tilde{q}_2'),$$
$$d_{\Sigma^2}((\alpha, \beta), (\alpha', \beta')) \leq \varepsilon + \max(\varepsilon_1, \varepsilon_2),$$
$$((\tilde{q}_1, \tilde{q}_2), (\tilde{q}_1', \tilde{q}_2')) \in \mathcal{R}.$$

Because $(q_i, q_i') \in \mathcal{R}_i$, $i = 1, 2$, we know that there exist $\alpha', \beta' \in \Sigma$ and $(\tilde{q}_1', \tilde{q}_2') \in Q_1' \times Q_2'$ such that

$$d_\Sigma(\alpha, \alpha') \leq \varepsilon_1, q_1' \overset{\alpha'}{\rightarrow}_{T_1'} \tilde{q}_1', (\tilde{q}_1, \tilde{q}_1') \in \mathcal{R}_1,$$
$$d_\Sigma(\beta, \beta') \leq \varepsilon_2, q_2' \overset{\beta'}{\rightarrow}_{T_2'} \tilde{q}_2', (\tilde{q}_2, \tilde{q}_2') \in \mathcal{R}_2.$$

It follows immediately that

$$((\tilde{q}_1, \tilde{q}_2), (\tilde{q}_1', \tilde{q}_2')) \in \mathcal{R}.$$

¿From the triangular inequality, we obtain

$$d_\Sigma(\alpha', \beta') \leq d_\Sigma(\alpha, \beta) + d_\Sigma(\alpha, \alpha') + d_\Sigma(\beta, \beta'),$$
$$\leq \varepsilon + \varepsilon_1 + \varepsilon_2,$$

and therefore $(q_1', q_2') \overset{\alpha',\beta'}{\rightarrow}_{T'} (\tilde{q}_1', \tilde{q}_2')$. Furthermore,

$$\max_{i \in \{\alpha,\beta\}} \max_{j \in \{\alpha',\beta'\}} d_\Sigma(i, j) \leq \varepsilon + \max(\varepsilon_1, \varepsilon_2).$$

Hence

$$d_{\Sigma^2}((\alpha, \beta), (\alpha', \beta')) \leq \max(\varepsilon_1, \varepsilon_2).$$

Finally, we need to show that for any $(q_1^0, q_2^0) \in Q_1^0 \times Q_2^0$ there exists $(q_1'^0, q_2'^0) \in Q_1'^0 \times Q_2'^0$ such that $\left((q_1^0, q_2^0), (q_1'^0, q_2'^0)\right) \in \mathcal{R}$. This fact is a direct consequence of $\mathcal{R}_1$ and $\mathcal{R}_2$ being the approximate simulation relations that define $T_1 \preceq_{\varepsilon_1,\delta_1} T_1'$ and $T_2 \preceq_{\varepsilon_2,\delta_2} T_2'$. ■

This result can be extended to approximate bisimulation, as follows.

Theorem 4.5: Given transition systems T_1, T_2, T_1' and T_2'. Suppose that the transition systems T_1 and T_1' have observation space Π_1, while T_2 and T_2' have observation space Π_2. Moreover we assume that all of them share the same set of labels Σ. If $T_1 \approx_{\varepsilon_1,\delta_1} T_1'$ and $T_2 \approx_{\varepsilon_2,\delta_2} T_2'$, then for any $\varepsilon \geq 0$,

$$T_1 \parallel_\varepsilon T_2 \approx_{\varepsilon+\max(\varepsilon_1,\varepsilon_2),\delta_1+\delta_2} T_1' \parallel_{\varepsilon+\varepsilon_1+\varepsilon_2} T_2'. \quad (24)$$

Proof: Analogous to that of Theorem 4.4. ■

Notice that when $\varepsilon = \varepsilon_1 = \varepsilon_2 = 0$, Theorem 4.4 and 4.5 are reduced to the already known compositionality properties of the approximate (bi)simulation relation in [11].

B. Composite transition systems

As explained in the previous subsection, the result of approximately synchronizing two transition systems is a kind of composite transition systems, whose transitions are labelled by a pair of labels. It is quite straightforward to generalize this idea, for example if we want to have several transition systems synchronizing. In this subsection, we formalize this idea and make it possible to discuss approximate synchronization of two (or more) composite transition systems.

Definition 4.6: Given a set of labels Σ, a **composite transition system** $T = (Q, \Sigma^n, \to, Q^0, \Pi, \langle\cdot\rangle)$ is a transition system with a set of labels Σ^n, $1 < n \in \mathbb{N}$. The number n is called the **multiplicity** of the composite transition systems.

Before we proceed to define approximate synchronization of composite transition systems (possibly with different multiplicities), we need to define a notion of distance between elements in Σ^n and Σ^m, where n and m are not necessarily equal.

Definition 4.7: Given $\sigma \in \Sigma^n$ and $\omega \in \Sigma^m$, we define the distance between σ and ω as

$$d_{\Sigma^*}(\sigma,\omega) = d_{\Sigma^*}(\omega,\sigma) := \max_{i=1,\dots,n} \max_{j=1,\dots,m} d_\Sigma(\sigma_i,\omega_j).$$

Definition 4.8: Given two composite transition systems $T_i = (Q_i, \Sigma^{n_i}, \to_i, Q_i^0, \Pi_i, \langle\cdot\rangle_i)$, $i = 1, 2$. The **approximate synchronization** operator $\parallel_\varepsilon$, $\varepsilon \geq 0$, acting on the two composite transition systems yield another composite transition system

$$T := T_1 \parallel_\varepsilon T_2, \quad (25)$$

where $T = (Q_1 \times Q_2, \Sigma^{n_1+n_2}, \to, Q_1^0 \times Q_2^0, \Pi_1 \times \Pi_2, \langle\cdot\rangle)$. The transition relation $\to$ is such that $(q_1, q_2) \overset{\sigma,\sigma'}{\to} (q_1', q_2')$ iff $q_1 \overset{\sigma}{\to}_1 q_1'$, $q_2 \overset{\sigma'}{\to}_2 q_2'$, $d_{\Sigma^*}(\sigma,\sigma') \leq \varepsilon$. The observation map $\langle\cdot\rangle$ is defined as

$$\langle(q_1, q_2)\rangle := (\langle q_1\rangle_1, \langle q_2\rangle_2). \quad (26)$$

The new observation space $\Pi = \Pi_1 \times \Pi_2$ is equipped with the pseudometric

$$d_\Pi((\pi_1,\pi_2),(\pi_1',\pi_2')) = d_{\Pi_1}(\pi_1,\pi_1') + d_{\Pi_2}(\pi_2,\pi_2').$$

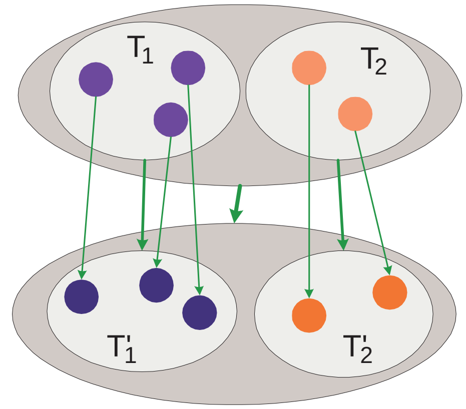

Fig. 2. Compositional properties of approximate (bi)simulation. Each ellipse symbolizes approximate synchronization. The arrows indicate approximate (bi)simulation. The relation between the precisions of the approximate (bi)simulations is given in Theorem 4.9 and not displayed here.

Notice that composite transition systems are intrinsically transition systems with an additional assumption in the structure of their sets of labels. Two composite transition systems with the same multiplicity share the same set of labels, and hence the concept of approximate (bi)simulation applies to them. The compositional properties of the approximate (bi)simulation in the previous subsection, which is defined for composite transition systems with multiplicity 2 can be extended easily to this more general case.

Theorem 4.9: Given a set of labels Σ and composite transition systems T_1, T_2, T_1' and T_2'. Suppose that the transition systems T_1 and T_1' have observation space Π_1 and multiplicity n_1, while T_2 and T_2' have observation space Π_2 and multiplicity n_2.

(i) If $T_1 \preceq_{\varepsilon_1,\delta_1} T_1'$ and $T_2 \preceq_{\varepsilon_2,\delta_2} T_2'$, then for any $\varepsilon \geq 0$,

$$T_1 \parallel_\varepsilon T_2 \preceq_{\varepsilon+\max(\varepsilon_1,\varepsilon_2),\delta_1+\delta_2} T_1' \parallel_{\varepsilon+\varepsilon_1+\varepsilon_2} T_2'. \quad (27)$$

(ii) If $T_1 \approx_{\varepsilon_1,\delta_1} T_1'$ and $T_2 \approx_{\varepsilon_2,\delta_2} T_2'$, then for any $\varepsilon \geq 0$,

$$T_1 \parallel_\varepsilon T_2 \approx_{\varepsilon+\max(\varepsilon_1,\varepsilon_2),\delta_1+\delta_2} T_1' \parallel_{\varepsilon+\varepsilon_1+\varepsilon_2} T_2'. \quad (28)$$

The compositional properties given in Theorem 4.9 is illustrated in Figure 2.

V. Conclusions

The notion of approximate (bi)simulation developed by Girard and Pappas [11], [12], [13] has developed as a useful tool of abstraction of dynamical systems. The theory stems from the idea of relaxing the requirement that an abstraction is exactly equal to the original system. In this paper, we follow the same path by imposing even more relaxed conditions on the approximate (bi)simulation. Namely, we introduce a pseudometric on the set of labels and allow some tolerance in the labels, when one system simulates another. We show that this new notion of approximate (bi)simulation is a generalization of the other one, in the sense that if we set the tolerance in the label to zero, we recover all the existing results.

Another notion that we introduce in this paper is that of approximate synchronization. Approximate synchronization is based on the idea of relaxing the requirements that when two transition systems synchronize, they synchronize on the same label. Instead, we allow them to synchronize on labels that are close. We show that approximate (bi)simulation is compositional with respect to approximate synchronization.

Having set up a theoretical framework, we set our next goal at providing a computational framework for the ideas that we discuss here. Approximate (bi)simulation of Girard and Pappas has a nice computational framework, in the form of bisimulation functions, to facilitate the construction of approximate (bi)simulation relations [12], [13], [17], [18]. We have generalized the notion of bisimulation function. We now need to extend the computation machinery to cope with the new notion.

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