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Stabilization and Tracking of Feedback Linearizable Systems under Input Constraints

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ABSTRACT

The problem of stabilization and trajectory tracking of feedback linearizable systems in the presence of input saturation is considered in this paper. The need to perform output tracking for nonlinear systems dictates the choice of feedback linearization in our control design. Explicit characterization of the space of feasible trajectories that can be tracked using bounded inputs is provided.

1. Introduction

In this paper, we are motivated in part by the successful application of feedback linearization techniques to *flight control* problems. In an aircraft there are inherent actuator constraints and thrust limitations which must be taken into account while planning the reference trajectory. This is an essential element of the automated *air traffic management system* (ATMS) proposed by NASA [1].

The problem of stabilization in the presence of input constraints goes as far back as [2]. In [3], it is shown that linear systems with poles in the right half plane can not be globally stabilized using bounded controls. In [4], it is shown that for chains of integrators with more than two states, one must use nonlinear feedback in order to obtain global stabilization. For null controllable systems one can obtain either global results using nonlinear control laws ([5]) or semi-global results using linear feedback ([6]).

Semi-global output regulation for linear, null controllable systems can be found in [7, 5]. For nonlinear systems, stabilization results for *feed-forward systems* can be found in [5], where the concept of nested saturation feedback is utilized.

In this paper, we analyze the effect of input saturation on feedback linearization and in designing controllers for stabilization and tracking. We calculate the regions of attraction of the controllers based on feedback linearization and characterize the space of feasible trajectories.

2. Stabilization

We consider the effect of input saturation on stabilization of nonlinear systems by first linearizing their input-output response. We assume that the reader is familiar with the fundamentals of feedback linearization at the level of [8].

Consider the SISO nonlinear system,

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \qquad (1)$$

with state $x \in R^n$, input $u \in R$, output $y \in R$, vector fields f, g on R^n, output function h,[1] and the constraint $|u| \leq M < \infty$. Without loss of generality, we assume that the origin is an equilibrium point of the autonomous system and that $h(0) = 0$. Let X be an open set containing the origin. When the system (1) has well defined relative degree γ, there exists a local diffeomorphism $\Phi : X \longrightarrow R^n$ such that the system (1) is expressed in the new coordinates as

$$\begin{aligned} \dot{z}_1 &= z_2 \\ &\vdots \\ \dot{z}_\gamma &= b(z,\eta) + a(z,\eta)u \\ \dot{\eta} &= q(z,\eta) \\ y &= z_1 \end{aligned} \qquad (2)$$

where $z = [z_1 \dots z_\gamma]$, η represent the zero dynamics, $b(z,\eta) = L_f^\gamma h(x)$ and $a(z,\eta) = L_g L_f^{\gamma-1} h(x)$. The feedback linearizing control law is

$$u = \frac{1}{L_g L_f^{\gamma-1} h(x)} (-L_f^\gamma h(x) + v) \qquad (3)$$

Since $|u| \leq M$ we must have

$$\left| \frac{1}{L_g L_f^{\gamma-1} h(x)} (-L_f^\gamma h(x) + v) \right| \leq M \qquad (4)$$

to linearize the system. From the above equation, it is clear that the cost of cancelling the nonlinearities can be large enough to saturate the control input. We therefore assume that the cost of feedback linearization does not saturate the control input, i.e.,

$$\left| \frac{L_f^\gamma h(x)}{L_g L_f^{\gamma-1} h(x)} \right| \leq M \qquad (5)$$

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[1] Vector fields f,g and function h are assumed to be as smooth as needed.

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Furthermore, in order to have some control authority remaining for stabilization purposes, we need,

$$\left|\frac{L_f^\gamma h(x)}{L_g L_f^{\gamma-1} h(x)}\right| \leq M - \varepsilon \tag{6}$$

for $0 < \varepsilon < M$. Notice that ε is essentially a parameter which determines the control authority distribution between the feedback linearizing cost and the stabilization cost. Equation (6) may not be satisfied for all $x \in X$. We will therefore restrict the state x to lie in the set $U_x \subset X \subset R^n$ defined by,

$$U_x = \{x \in R^n : \left|\frac{L_f^\gamma h(x)}{L_g L_f^{\gamma-1} h(x)}\right| \leq M - \varepsilon\} \tag{7}$$

which, as a result, restricts (z, η) to lie in $U_{z\eta} = \Phi(U_x) \subset Z$. We must, therefore, guarantee that during the stabilization process, the state x (equivalently (z, η)) never leaves the set U_x ($U_{z\eta}$) so that the control input is not saturated due to the control cost of feedback linearization. From (4) we obtain,

$$\begin{aligned} |u| &= \left|-\frac{L_f^\gamma h(x)}{L_g L_f^{\gamma-1} h(x)} + \frac{v}{L_g L_f^{\gamma-1} h(x)}\right| \\ &\leq M - \varepsilon + \left|\frac{v}{L_g L_f^{\gamma-1} h(x)}\right| \end{aligned}$$

Therefore, if v satisfies,

$$\left|\frac{v}{L_g L_f^{\gamma-1} h(x)}\right| \leq \varepsilon \Longrightarrow |v| \leq \varepsilon \, |L_g L_f^{\gamma-1} h(x)|$$

then, we can guarantee that $|u| \leq M$. Since the system has relative degree γ, we have $L_g L_f^{\gamma-1} h(x) \neq 0$ in an open neighborhood of the origin, and without loss of generality we will assume from now on that $L_g L_f^{n-1} h(x) > 0$ for $x \in U_x$. As a result, the external input v is constrained in the set $V(x)$ defined as

$$V(x) = [-\varepsilon L_g L_f^{\gamma-1} h(x), \; \varepsilon L_g L_f^{\gamma-1} h(x)] \tag{8}$$

Since $L_g L_f^{\gamma-1} h(x) > 0$ for $x \in U_x$, we define

$$L = \inf_{x \in U_x} L_g L_f^{\gamma-1} h(x)$$

Clearly, L always exists but it may be zero. If $L = 0$ then the previous analysis can be repeated in a smaller, compact neighborhood of 0 in which case $L > 0$. We therefore assume that $L > 0$. Define

$$\bar{V} = [-\varepsilon L, \varepsilon L]$$

It is obvious that $\bar{V} \subset V(x)$. Thus, we avoid the difficulty of dealing with a state dependent set, namely $V(x)$ by working with the more conservative, but state independent set $\bar{V}$.

We are now faced with the problem of stabilizing $(z(0), \eta(0)) \in U_{z\eta}$ to the origin of the system,

$$\begin{aligned} \dot{z}_1 &= z_2 \\ &\vdots \\ \dot{z}_\gamma &= v \\ \dot{\eta} &= q(\eta, z) \\ y &= z_1 \end{aligned}$$

under the control constraint $v \in \bar{V}$ and the state constraint that $(z(t), \eta(t)) \in U_{z\eta}$ for all time t.

Theorem 1 *Let $s^\gamma + c_{\gamma-1} s^{\gamma-1} + \ldots + c_1 s + c_0$ be a Hurwitz polynomial and let $c = [c_0 c_1 \ldots c_{\gamma-2} c_{\gamma-1}]^T$. If the zero dynamics satisfy for some $\bar{\beta}, \bar{\delta}, \mu > 0$,*

$$\|\eta(t)\|_2 < \bar{\beta} \, \|\eta(0)\|_2 \, e^{-\mu t} + \bar{\delta} \, \|z(t)\|_2 \tag{9}$$

then, there exist $r_z, r_\eta > 0$ such that for all initial conditions satisfying $\|z(0)\|_2 \leq r_z$ and $\|\eta(0)\|_2 \leq r_\eta$, the following linear control law

$$v = -c_0 z_1 - c_1 z_2 - \ldots - c_{n-1} z_n \tag{10}$$

will result in $(z(t), \eta(t))$ converging to the origin exponentially while satisfying the control constraint $v \in \bar{V}$ and state constraint $(z(t), \eta(t)) \in U_{z\eta}$ for all time t.

Proof: Let r be the radius of the largest ball, call it $B_{z\eta}$, in $U_{z\eta}$. In order to guarantee that the state constraint is satisfied it suffices to show that $(z(t), \eta(t)) \in B_{z\eta} \subset U_{z\eta}$. Since the polynomial $s^n + c_{n-1} s^{n-1} + \ldots + c_1 s + c_0$ is Hurwitz then the following control

$$v = -c^T z \tag{11}$$

will ensure that, in the absence of the state and control constraints, there exist $K, \lambda > 0$ such that

$$\|z(t)\| \leq K \|z(0)\| e^{-\lambda t}$$

Therefore,

$$\|z(t)\| \leq K \|z(0)\|$$

Let $\bar{r}_z$ be the radius of a ball, B_z, which is to be determined. Then if we choose $r_z = \frac{\bar{r}_z}{K}$, we obtain

$$\|z(t)\| \leq K \|z(0)\| \leq \bar{r}_z$$

and therefore $z(t) \in B_z$. However, since we want $v \in \bar{V}$ we must impose the condition,

$$|c^T z| \leq \varepsilon L \tag{12}$$

By the Cauchy-Schwartz inequality we obtain,

$$\|c\|_2 \, \|z\|_2 \leq \varepsilon L \tag{13}$$

Condition (13) reflects the natural tradeoff between the rate of convergence and the domain of attraction. This implies that

$$\|z(t)\|_2 \leq \frac{\varepsilon L}{\|c\|_2} \tag{14}$$

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This is true if $z(0)$ satisfies,

$$\|z(0)\|_2 \leq \frac{\varepsilon L}{\|c\|_2 K} \tag{15}$$

Therefore, if we choose $r_z = \min(\frac{\varepsilon L}{\|c\|_2 K}, \frac{\bar{r}_z}{K})$ then for all initial conditions $\|z(0)\| \leq r_z$, $z(t) \longrightarrow 0$ and $z(t) \in B_z$ while satisfying the control constraint.

The problem now reduces to finding $\bar{r}_z$ such that the state constraint is satisfied. In addition to $z(t) \rightarrow 0$, we also require stabilization of the η dynamics in order to stabilize $x(t)$ to the equilibrium point $(x = 0) = \Phi^{-1}(0,0)$. Thus, we need to restrict the behavior of the η dynamics.

Imposing the condition that the system (1) be exponentially minimum phase will guarantee that the origin of the η subsystem

$$\dot{\eta} = q(\eta, 0)$$

is locally exponentially stable. However in our case we have

$$\dot{\eta} = q(\eta, z) \tag{16}$$

We can think of z as an input to the system (16). Therefore depending on $z(t)$, the η dynamics may grow to a point where they knock us out of the desirable region, namely $U_{z\eta}$.

In order to remedy this situation, we require the system (16) to be Input-to-State-Stable (ISS).

We use the notion of input-to-state stability in a slightly stronger form. We require that the system (16) satisfies the following finite gain ISS condition:

$$\|\eta(t)\|_2 < \bar{\beta}\, \|\eta(0)\|_2\, e^{-\mu t} + \bar{\delta}\, \|z(t)\|_2 \tag{17}$$

for some $\mu, \bar{\beta}, \bar{\delta} > 0$. If we now choose $\bar{r}_z$ and r_η to satisfy the following condition,

$$(1 + \bar{\delta})\, \bar{r}_z + \bar{\beta}\, r_\eta \leq r \tag{18}$$

then we get,

$$\begin{aligned} \left\| \begin{bmatrix} z(t) \\ \eta(t) \end{bmatrix} \right\|_2 &\leq \|z(t)\|_2 + \|\eta(t)\|_2 \\ &\leq (1 + \bar{\delta})\, \|z(t)\|_2 + \bar{\beta}\, \|\eta(0)\|_2\, e^{-\mu t} \\ &\leq r \end{aligned}$$

and therefore the state trajectory will stay in the desired set $B_{z\eta} \subset U_{z\eta}$ and as a result the state constraint is also satisfied. □

A simplified version of this theorem in the case where the system is full state feedback linearizable can be easily obtained as a corollary [9].

2.1 Nonlinear Stabilization

We now take a different approach to stabilization using feedback linearization. The feedback linearizing law,

$$u = \frac{1}{L_g L_f^{n-1} h(x)} \left[-L_f^n h(x) + v \right]$$

along with the input constraint $|u| \leq M$, implies that v must satisfy,

$$\begin{aligned} -M \cdot L_g L_f^{n-1} h(x) + L_f^n h(x) &\leq v \\ v &\leq M \cdot L_g L_f^{n-1} h(x) + L_f^n h(x) \end{aligned} \tag{19}$$

In the last section, we assumed strict bounds on the nonlinearities given by equation (6). This approach resulted in conservative solutions, whereas the real constraint on input v is given by equations (19). Thus, in this case, the external input v can, in some sense, *help* the input u not to saturate. Thus, the question now shifts to the following: Given

$$\begin{aligned} \dot{z}_1 &= z_2 \\ &\vdots \\ \dot{z}_n &= v \end{aligned} \tag{20}$$

with $v \in [f_1(x), f_2(x)]$, find conditions on $f_1(x), f_2(x)$ so that the system (20) is stabilizable. Notice that placing conditions on $f_1(x)$ and $f_2(x)$ will give rise to state constraints as in previous section. We will investigate it for simple systems. Consider

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= v \end{aligned}$$

If we use

$$v = -k_2\, \mathrm{sgn}[x_2 - \{-k_1\, \mathrm{sgn}\,(x_1)\}]$$

with $k_2 > k_1 > 0$ and if $f_1(x) \leq -k_2$ when $v < 0$ and $f_2(x) \geq k_2$ when $v > 0$ then the system will be globally stable. It is evident that, this is not a necessary condition for stabilization as it depends on the choice of the control law for v.

Generalizations of this scheme to more than two dimensions can be easily worked out. This approach of using nested saturation functions was first introduced in [5] in the case of state independent control constraints. This approach cannot be directly applied to the problem of feedback linearization under constraints because it is difficult to show that the state constraints will be satisfied.

3. Trajectory Tracking

In the presence of control constraints one is interested in knowing what is the space of trajectories that are feasible for both exact and asymptotic tracking. Reconsider system (2). From now on we assume that the system is full state linearizable. We wish that $y(t)$ is exactly equal to some desired trajectory $y_d(t) \in C^n(R_+)$. Let $z_d(t) = [y_d(t)\ y_d^{(1)}(t)\ \ldots y_d^{(n-1)}(t)]^T$. Exact output tracking requires that $z(0) = z_d(0)$ and that the control input (3) is

$$u = \frac{1}{a(z)}(-b(z) + v) = \frac{1}{a(z_d)}(-b(z_d) + y_d^{(n)}) \tag{21}$$

Now since $|u| \leq M$ we must have that

$$\left| \frac{y_d^{(n)} - b(z_d)}{a(z_d)} \right| \leq M \tag{22}$$

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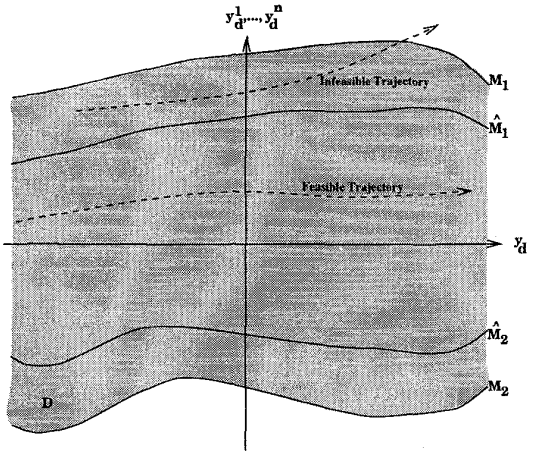


Figure 1: Space of Feasible Trajectories for Exact and Asymptotic Tracking

Consider the map $F : C^n(R_+) \longrightarrow R$ defined by

$$F : y_d(t) \longrightarrow \sup_{t \in R_+} \left| \frac{y_d^{(n)} - b(y_d, y_d^{(1)}, \ldots, y_d^{(n-1)})}{a(y_d, y_d^{(1)}, \ldots, y_d^{(n-1)})} \right| \quad (23)$$

which given a trajectory $y_d(t)$ simply returns the maximum value of control input u it will require. If $C^n(R_+)$ is equipped with the uniform topology and R has the standard topology then F is a continuous map since it is the composition of continuous functions. Therefore the space of feasible trajectories is

$$Y = F^{-1}([-M, M]) \quad (24)$$

and is therefore a closed set in $C^n(R_+)$. Furthermore, unless $M = 0$, Y is a rich set of trajectories. Feasible trajectories which will saturate the input u must satisfy $F(y_d(t)) = M$ or otherwise

$$\left| \frac{y_d^{(n)} - b(y_d, y_d^{(1)}, \ldots, y_d^{(n-1)})}{a(y_d, y_d^{(1)}, \ldots, y_d^{(n-1)})} \right| = M \quad (25)$$

Define the maps $F_1, F_2 : R^{n+1} \longrightarrow R$ by

$$\begin{aligned} F_1 &= y_d^{(n)} - b(y_d, \ldots, y_d^{(n-1)}) - M\, a(y_d, \ldots, y_d^{(n-1)}) \\ F_2 &= y_d^{(n)} - b(y_d, \ldots, y_d^{(n-1)}) + M\, a(y_d, \ldots, y_d^{(n-1)}) \end{aligned}$$

and consider the sets:

$$M_1 = \{(y_d, \ldots, y_d^{(n)}) \in R^{n+1} : F_1 = 0\} \quad (26)$$

$$M_2 = \{(y_d, \ldots, y_d^{(n)}) \in R^{n+1} : F_2 = 0\} \quad (27)$$

It is easy to show that M_1 and M_2 are manifolds of co-dimension 1 in R^{n+1}. Also, provided $M \neq 0$ and $|a(y_d, y_d^{(1)}, \ldots, y_d^{(n-1)})| \geq L > 0$, M_1 and M_2 do not intersect and therefore they define a submanifold D of R^{n+1} of dimension $n+1$ with boundary $M_1 \cup M_2$ as shown in Figure 1.

The space of feasible trajectories, Y, can now be thought of as the space of all curves in R^{n+1} whose image lies in D.

For asymptotic tracking, we again have the chain of integrators (2) but in this case we have $z(0) \neq z_d(0)$. Therefore, the following change of coordinates $e = z - z_d$ results in

$$\begin{aligned} \dot{e}_1 &= e_2 \\ &\vdots \\ \dot{e}_n &= b(z) + a(z)u - y_d^{(n)} \end{aligned} \quad (28)$$

If $s^n + c_{n-1}s^{n-1} + \ldots + c_1 s + c_0$ is a Hurwitz polynomial then the following control law

$$u = \frac{y_d^{(n)} - b(z) - c_0 e_1 - c_1 e_2 - \ldots - c_{n-1} e_n}{a(z)} \quad (29)$$

will ensure that the error e converges to zero exponentially if the input bounds are not violated. Since the input u is bounded, we must have

$$|u| = \left| \frac{y_d^{(n)} - b(e + z_d) - c^T e}{a(e + z_d)} \right| \leq M \quad (30)$$

By assumption, f, g and h are smooth and therefore the functions $a(z)$ and $b(z)$ are locally Lipschitz continuous. Therefore, we can find constants k_a and k_b (truncating the neighborhood U_x if necessary) such that:

$$|a(z) - a(\tilde{z})| \leq k_a \|z - \tilde{z}\|_2 \quad (31)$$

$$|b(z) - b(\tilde{z})| \leq k_b \|z - \tilde{z}\|_2 \quad (32)$$

Under this assumption, simple analysis (see [9]) shows that a sufficient condition for $|u| \leq M$ is:

$$\left| \frac{y_d^{(n)} - b(z_d)}{a(z_d)} \right| \leq \frac{M}{1 + \frac{k_a \|e\|_2}{L}} - \frac{(k_b + \|c\|_2)\|e\|_2}{L} \quad (33)$$

The norm of the tracking error for any time can be bounded by the following expression:

$$\|e(t)\|_2 \leq K \|e(0)\|_2 \quad (34)$$

Define:

$$\tilde{M} = \frac{M}{1 + \frac{k_a K \|e(0)\|_2}{L}} - \frac{(k_b + \|c\|_2) K \|e(0)\|_2}{L} < M$$

Condition 33 is very similar to condition (22) for exact tracking, with M replaced by $\tilde{M}$. In fact the two conditions become the same for $\|e(0)\|_2 = 0$, as expected. Thus the argument presented above about the feasible trajectories lying between two manifolds in R^{n+1} is still valid, with the manifolds M_1 and M_2 replaced by manifolds, $\hat{M}_1$ and $\hat{M}_2$, which are "parallel" to the original and lie inside the region D (see Figure 1). The only requirement for a nontrivial set of feasible trajectories is that $\tilde{M} > 0$. Because $\tilde{M} \to M > 0$ as $\|e(0)\|_2 \to 0$ it seems plausible that, if we start sufficiently close to the desired trajectory (or rather the desired trajectory starts sufficiently close to the current system state), $\tilde{M} > 0$.

Lemma 1 *There exists $r > 0$ such that $\|e(0)\|_2 < r$ implies $\tilde{M} > 0$.*

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Proof: In [9].

The following theorem, provides sufficient conditions under which the tracking error will exponentially converge to zero.

Theorem 2 *Consider system (2) and let two constants $\varepsilon > \varepsilon_1 > 0$ be given. If the reference trajectory satisfies the following conditions,*

$$\left|\frac{b(y_d, y_d^{(1)}, \ldots, y_d^{(n-1)})}{a(y_d, y_d^{(1)}, \ldots, y_d^{(n-1)})}\right| \leq M - \varepsilon - \varepsilon_1 \tag{35}$$

$$\left|\frac{y_d^{(n)}}{a(z_d)}\right| \leq \varepsilon - \varepsilon_1 - \frac{Mk_a\|e\|_2}{L + k_a\|e\|_2} - \frac{k_b\|e\|_2}{L} \tag{36}$$

the initial error lies in a ball of radius $\frac{r_e}{K}$ where K is defined as in equation (34) and r_e satisfies,

$$\frac{k_a k_b}{L^2} r_e^2 + \frac{k_b + k_a(M - \varepsilon - \varepsilon_1)}{L} r_e \leq \varepsilon_1 \tag{37}$$

and the Hurwitz polynomial $s^n + c_{n-1}s^{n-1} + \ldots + c_0$ satisfies

$$\|c\|_2 \leq \frac{\varepsilon_1 L}{r_e} \tag{38}$$

then the system will asymptotically track the desired reference trajectory while satisfying the control and state constraints.

Proof: If it happens that for some values of r_e that satisfy equation (37), the right hand side of equation (36) is negative then choose a smaller value of r_e for which (36) results in a nontrivial set of trajectories. Now since the polynomial is Hurwitz we know that the error will exponentially converge to zero and furthermore since $\|e(0)\| \leq \frac{r_e}{K}$ we have $\|e(t)\| \leq r_e$ for all time t. We are thus left to show that we will satisfy the control and state constraints. To prove that we satisfy the state constraint we must show that $z(t) \in U_z$ or otherwise to show that

$$\left|\frac{b(z)}{a(z)}\right| \leq M - \varepsilon \tag{39}$$

Some algebra shows that

$$\begin{aligned}\left|\frac{b(z)}{a(z)}\right| &= \left|\frac{b(z_d + e)}{a(z_d + e)}\right| \\ &\leq \left|\frac{k_b\|e\|_2 + b(z_d)}{a(z_d)}\right| \left(1 + \frac{k_a\|e\|_2}{L}\right) \\ &\leq M - \varepsilon\end{aligned}$$

and therefore the state constraint is satisfied. Similarly, after some calculations

$$\begin{aligned}|u| &\leq \left(M - \varepsilon_1 - \frac{Mk_a\|e\|_2}{L + k_a\|e\|_2}\right)\left(1 + \frac{k_a\|e\|_2}{L}\right) \\ &\leq M\end{aligned}$$

and the control constraint is also satisfied. □

Theorem 2 can be applied in the case of exact tracking as corollary and can be extended in the case where zero dynamics are present. These results can be found in [9].

3.1 Extension to MIMO systems

Consider the square MIMO nonlinear control system described by

$$\begin{aligned}\dot{x} &= f(x) + g(x)u \\ y &= h(x)\end{aligned} \tag{40}$$

Under the assumption that the system has a well defined *vector relative degree*[8], we have,

$$y = b(x) + A(x)\,u \tag{41}$$

$A(x)$ is called the decoupling matrix. If $A(x)$ is locally invertible, then the *static state feedback* given by

$$u = (A(x))^{-1}\,[\,-b(x) + v\,] \tag{42}$$

will result in a closed loop system which is decoupled from inputs v to outputs y.

In case of input saturation, we have to add the following constraints to the system equations (40):

$$|u_i| \leq M_i \qquad M_i > 0, \quad i = 1, \cdots, m \tag{43}$$

If we define the vector $B(x)$ as

$$\begin{bmatrix} B_1(x) \\ \vdots \\ B_m(x) \end{bmatrix} = (A(x))^{-1} \begin{bmatrix} b_1(x) \\ \vdots \\ b_m(x) \end{bmatrix} \tag{44}$$

then the feedback linearizing control law (42) has to satisfy

$$|B_i(x) + (A(x))^{-1}v_i| \leq M_i \tag{45}$$

for $1 \leq i \leq m$ and the MIMO problem has been converted into m decoupled SISO problems with state dependent input constraints. Although, this looks straight forward, we will point out various subtleties involved in carrying out these calculations.

1. The constraints on u_i get transferred to those on v_j depending on the matrix $(A(x))^{-1}$. Thus, if the matrix $A(x)$ is numerically ill conditioned, solutions will be very conservative. This can be avoided, only if the quantities $(A(x))^{-1}$ and $b(x)$ can be computed analytically.
2. In the case of $A(x_0)$ being singular, one has to resort to dynamic state feedback. All the algorithms for dynamic state feedback linearization (e.g.,[8]) involve adding integrators in front of the inputs, thereby making it even more difficult to transform the input constraints from u_i to v_j.
3. Another MIMO problem which is not encountered in SISO systems is that of internal dynamics. Except in the very rare circumstances when the distribution $\Delta(x) = \text{span}[g_1, \cdots, g_m](x)$ is involutive (see [8]), the equation for the η dynamics depends explicitly on the input u.

$$\dot{\eta} = Q(z, \eta) + P(z, \eta)u$$

Looking at the analysis of the previous section, the explicit dependence of $\eta(t)$ on z and u will restrict the space of trajectories even further.

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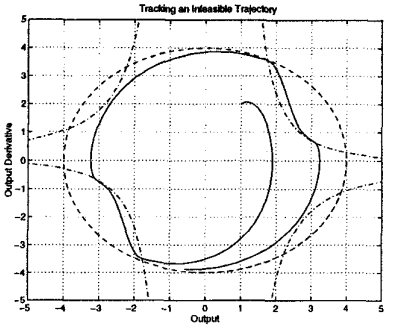


Figure 2: Van der Pol Oscillator Tracking Sinusoids

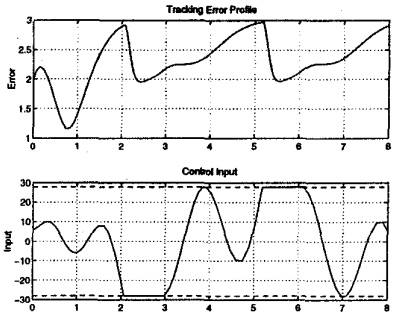


Figure 3: Tracking Error and Control Input

4. Simulation Results

The results of the previous sections are now applied on the controlled Van der Pol oscillator whose equations are given by ([8]),

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= 2\omega\zeta(1-x_1^2)x_2 - \omega^2 x_1 + u \\ y &= x_1 \end{aligned}$$

where ω is the frequency of oscillation and ζ is the damping factor. Simple calculations show that, the feedback linearizing law is

$$u = -2\omega\zeta(1-x_1^2)x_2 + \omega^2 x_1 + v$$

In order to track a reference trajectory $y_d(t)$ we use,

$$v = \ddot{y}_d(t) - c_1(x_1 - y_d(t)) - c_2(x_2 - \dot{y}_d(t))$$

The control input must satisfy $|u| \leq M$ and thus we define the set $U_x = U_z$ as,

$$U_x = \{x \in R^2 : |-2\omega\zeta(1-x_1^2)x_2 + \omega^2 x_1| \leq M - \varepsilon\}$$

The oscillator is asked to track the reference trajectory, $y_d(t) = A\cos t$. Parameter values were $M = 28$, $\omega = 2$,$\zeta = 0.7$, $A = 1$, $c_1 = 3$,$c_2 = 2$ and the initial condition was $[2,2]^T$. For these values the control does not saturate and tracking is performed.

Increasing the amplitude of the desired sinusoid to $A = 4$ results in the desired trajectory crossing out of U_x (dash-dot lines), as shown in Figure 2. Therefore, this sinusoid is not feasible. The initial condition $[1,2]^T$ results in the state trajectory of Figure 2. Notice that initially, since the state lies in U_x, the actual trajectory converges to the desired one. However, once the state trajectory crosses out of U_x, the cost of feedback linearization is too high and the input saturates as shown in Figure 3. The output then diverges from the desired trajectory. Once the desired trajectory re-enters U_x there is some control authority available for tracking purposes and the system is again tracking until the sinusoid leaves U_x. This story repeats itself as can seen from Figure 3 which displays the error profile and the saturating input.

5. Conclusions

In this paper, we have analyzed the feedback linearization framework in the presence of input constraints. Feedback linearization was used in order to perform trajectory tracking for nonlinear systems. Sufficient conditions for both local stabilization and trajectory tracking as well as characterizations of the space of feasible trajectories were provided. Simulations verified theoretical derivations.

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