APPROXIMATION OF NETWORK LINEAR OPERATORS USING SIMILARITY SHIFT FILTERS

Cassiano O. Becker, Sérgio Pequito, George J. Pappas, and Victor M. Preciado

University of Pennsylvania
Deparment of Electrical and Systems Engineering
Philadelphia, PA 19104 USA

ABSTRACT

We address the problem of designing linear filters based on polynomial shift operators to implement network linear transformations. We introduce the *similarity shift* and the associated *similarity shift filter*, and find conditions for perfect reconstruction of the desired transformation. In addition, when such reconstruction is not possible, we provide an approximation, and supplement its analysis with the quantification of the error incurred. To compare the studied scenarios with existing approaches, we perform numerical simulations in two distinct cases: Erdős-Rényi random networks and brain networks based on a real dataset from neuroimaging. For a given performance level, we observed that the similarity shift method was able to approximate target operators with filters with comparatively lower degree. This characteristic allows linear operators with larger dimensions to be considered in practice.

Index Terms— signal processing, signal reconstruction, linear filters, graph signal processing, similarity transformation

I. INTRODUCTION

Signal processing provides a principled and powerful approach to deal with the increasing information demands that are associated with modern data infrastructure. As part of its methodology, a fundamental operation is the one of *filtering*, which consists of applying a prescribed transformation to a signal of interest [1]. More specifically, in the important class of linear filters, such transformation is achieved by successively applying a *shift* operator on a signal, where the effect of each iteration is weighted by specific filter coefficients. With that respect, this paper considers the *linear filter design problem*, which consists of finding appropriate filter coefficients for a parametrized filter, given by a polynomial description of shift operators, such that the desired linear transformation of the signal (referred to as the *target* operator) is achieved [2, 3].

The theory of signal processing has benefited greatly from the formalization introduced by algebraic signal processing [4, 5], which expanded the class of shift operators that can be considered. In that respect, one important extension was the framework of graph signal processing [6, 7, 8]. In graph signal processing, each dimension of a signal is identified with nodes of an underlying graph, and the a shift operator can capture, for instance, relational dependencies among the data. This approach allows one to study processes on networks from the standpoint of graph signals and graph filters acting on a network, which is in turn described by a graph shift operator (see for instance [9]).

The main contributions of this paper are as follows. First, we introduce a linear filter design approach based on a generalization of shift operators. By applying a specific transformation on the shift operator, we are able to surmount constraints typically imposed on the joint spectral characteristics of the target and shift operators. As a result, our method achieves specified performance levels within comparatively fewer iterations, making it suitable for signals and networks of larger dimensions.

The rest of the paper is structured as follows: in Section II, we develop the mathematical preliminaries; in Section III, we describe our main results; in Section IV, we perform numerical experiments on both simulated networks and on a real neuroimaging dataset; and in Section V we conclude with closing remarks.

II. PRELIMINARIES

In this paper, we adopt common terminology and notation[1]. More specifically, we define the signal $\mathbf{x}$ as an element of a vector space of finite dimension N over the base field $\mathbb{C}$, i.e., $\mathbf{x} \in \mathbb{C}^N$. A linear operator $\mathcal{B} : \mathbb{C}^N \to \mathbb{C}^N$ is represented by a matrix $\mathbf{B} \in \mathbb{C}^{N\times N}$ and acts on a signal $\mathbf{x}$ producing an output $\mathbf{y} = \mathbf{B}\mathbf{x}$. Considering a matrix $\mathbf{S} \in \mathbb{C}^{N\times N}$ as the shift operator, we define a linear filter $\mathbf{H}$ as a matrix polynomial $h : \mathbb{C}^{N\times N} \to \mathbb{C}^{N\times N}$ of order $L-1$, such that $\mathbf{H} := h(\mathbf{S}) := \sum_{l=0}^{L-1} c_l \mathbf{S}^l$. We assume the shift $\mathbf{S}$ to be a diagonalizable matrix and let $\mathbf{S} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1} = \mathbf{V}\text{diag}(\boldsymbol{\lambda})\mathbf{V}^{-1}$. Since the matrix $\mathbf{V}$ diagonalizes the shift operator, we refer to it as the Fourier basis of $\mathbf{S}$ [6], and denote the Fourier transform of the signal $\mathbf{x}$ with respect to the shift $\mathbf{S}$ as $\hat{\mathbf{x}}_{\mathbf{S}} = \mathbf{V}^{-1}\mathbf{x}$. Expressing the filter $\mathbf{H}$ in the Fourier basis of $\mathbf{S}$ we have

$$\begin{aligned}\mathbf{H} &= \sum_{l=0}^{L-1} c_l \mathbf{S}^l = \sum_{l=0}^{L-1} c_l (\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1})^l = \sum_{l=0}^{L-1} c_l \mathbf{V}\mathbf{\Lambda}^l\mathbf{V}^{-1} \\ &= \mathbf{V}\Big(\sum_{l=0}^{L-1} c_l \mathbf{\Lambda}^l\Big)\mathbf{V}^{-1} = \mathbf{V}h(\mathbf{\Lambda})\mathbf{V}^{-1},\end{aligned}$$

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[1]**Notation**: We denote by $\mathbf{x}$ a column vector in $\mathbb{C}^n$ and by x_i its i-th entry. Bold capital letters are reserved for matrices, for instance $\mathbf{X} \in \mathbb{C}^{m\times n}$, where $[\mathbf{X}]_{ij}$ denotes the entry in the i-th row and j-th column of $\mathbf{X}$. An indexed vector $\mathbf{x}_i$ (respectively matrix $\mathbf{X}_i$) denotes an element in an indexed set of vectors (respectively, matrices). The $n \times n$ identity matrix is denoted by $\mathbf{I}_n = \text{diag}(\mathbf{1}_n)$, where $\text{diag}(\mathbf{x})$ is a diagonal matrix having the entries of $\mathbf{x}$ in its diagonal. The canonical vector $\mathbf{e}_k$ has all entries equal to zero except at its k-th position, which is set to 1. The pseudo-inverse of a matrix $\mathbf{A}$ is denoted by $\mathbf{A}^\dagger$.

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and refer to $h(\mathbf{\Lambda})$ as the *frequency response* of the filter $\mathbf{H}$. In addition, denoting by $\boldsymbol{\lambda}^k$ the vector containing the eigenvalues of $\mathbf{S}$ to the k-th power, i.e., $[\lambda_1^k, \ldots, \lambda_N^k]^T$, we can define the Vandermonde matrix $\mathbf{\Psi} \in \mathbb{C}^{N\times L}$ as the matrix $\mathbf{\Psi} := [\boldsymbol{\lambda}^0, \ldots, \boldsymbol{\lambda}^{L-1}]$. Using the Vandermonde matrix with $L = N$ and building a vector $\mathbf{c} = [c_0, \ldots, c_{N-1}]$ with the elements of the set of filter coefficients, we may write the frequency response of the filter $\mathbf{H}$ as $h(\mathbf{\Lambda}) = \text{diag}(\mathbf{\Psi c})$. Finally, we can associate the shift operator $\mathbf{S}$ with an underlying directed graph $\mathcal{G}$, whose set of nodes we denote by $\mathcal{N}$. The corresponding set of links is denoted by $\mathcal{E} \subseteq \mathcal{N}\times\mathcal{N}$, such that $[\mathbf{S}]_{ij}$ is non-zero only if $i = j$ or $(i, j) \in \mathcal{E}$. Then $\mathbf{S}$ can be referred to as a *graph-shift* operator, and a filter using the graph-shift operator is referred to as a *graph-filter*. In this case, the signal $\mathbf{x}$ can be interpreted as a function $f : \mathcal{N} \to \mathbb{R}$ defined on the vertices of the graph.

III. Reconstruction using Similarity Filters

In this section, we develop the results for perfect and approximate reconstruction of a linear operator $\mathbf{B}$ through a linear filter $\mathbf{H}$. By perfect implementation we refer to a filter that achieves $\mathbf{y} = \mathbf{Hx} = \mathbf{Bx}$ for all signals $\mathbf{x}$ (see Theorem 1). In the case of approximate implementation, we aim at determining an approximating filter $\tilde{\mathbf{H}}$ that minimizes the error $\|\tilde{\mathbf{H}}\mathbf{x} - \mathbf{Bx}\|_2$ for all $\mathbf{x}$, see Theorem 2. To achieve these goals we start by introducing the *similarity filter* as follows:

$$\mathbf{H}_s := h_s(\mathbf{A}(\mathbf{S})) = \sum_{l=0}^{L-1} c_l \left(\mathbf{A}(\mathbf{S})\right)^l, \tag{1}$$

which is based on the *similarity* shift $\mathbf{A}(\mathbf{S})$. The similarity shift $\mathbf{A}$ is associated with the *base* shift $\mathbf{S}$ through

$$\mathbf{A}(\mathbf{S}) := \mathbf{TST}^{-1}, \tag{2}$$

which is a similarity transformation of $\mathbf{S}$, parameterized by the *similarity transformation matrix* $\mathbf{T}$. Notice that the similarity filter $\mathbf{H}_s$ is *shift-invariant* [4], since it fulfills $\mathbf{H}_s\mathbf{Ax} = \mathbf{AH}_s\mathbf{x}$.

The filter design problem with respect similarity shifts can be formally described as follows: given a target linear operator $\mathbf{B}$ and a base shift $\mathbf{S}$, find the filter coefficients $\{c_l\}_{l=0}^{L-1}$ and a similarity transformation matrix $\mathbf{T}$ to achieve perfect, or minimum error approximate implementation. In order to address this problem, we begin by looking at properties of the proposed filter. First, note that we can express the similarity filter in terms of the original graph filter as

$$\mathbf{H}_s := \sum_{l=0}^{L-1} c_l(\mathbf{TST}^{-1})^l = \mathbf{T}(\sum_{l=0}^{L-1} c_l\mathbf{S}^l)\mathbf{T}^{-1}. \tag{3}$$

Furthermore, if we perform an eigendecomposition of $\mathbf{S}$, we have

$$\begin{aligned}\mathbf{H}_s &= \mathbf{T}(\sum_{l=0}^{L-1} c_l\mathbf{V}\mathbf{\Lambda}^l\mathbf{V}^{-1})\mathbf{T}^{-1} \\ &= \mathbf{TV}(\sum_{l=0}^{L-1} c_l\mathbf{\Lambda}^l)\mathbf{V}^{-1}\mathbf{T}^{-1} = \mathbf{TV}h(\mathbf{\Lambda})\mathbf{V}^{-1}\mathbf{T}^{-1}.\end{aligned}$$

Note that, for a common set of coefficients $\{c_l\}_{l=0}^{L-1}$, the similarity shift $\mathbf{A}$ induces the same frequency response $h(\mathbf{\Lambda})$ as the one obtained for the shift $\mathbf{S}$, however on a different Fourier basis. This fact is leveraged in the theorem described next, where we find conditions for perfect reconstruction of the target operator $\mathbf{B}$ by the linear filter $\mathbf{H}_s$.

Theorem 1. (Perfect reconstruction) *Let $\mathbf{B} \in \mathbb{C}^{N\times N}$ be a target linear operator, and $\mathbf{S} \in \mathbb{C}^{N\times N}$ a base shift matrix. In addition, let $\mathbf{A} \in \mathbb{C}^{N\times N}$ be a similarity shift parametrized by the similarity transformation matrix $\mathbf{T} \in \mathbb{C}^N$, and $\mathbf{H}_s$ the linear filter defined in* (1)*, with $\mathbf{c} \in \mathbb{C}^N$ its vector of filter coefficients. Then, $\mathbf{H}_s$ implements the linear transformation $\mathbf{B}$ if the following conditions are satisfied:*

(i) matrices $\mathbf{S}$ and $\mathbf{B}$ are diagonalizable; and
(ii) matrix $\mathbf{S}$ has all distinct eigenvalues. ◇

Proof. Since $\mathbf{S}$ and $\mathbf{B}$ are diagonalizable by (*i*), we can write $\mathbf{S} = \mathbf{V\Lambda V}^{-1}$, and $\mathbf{B} = \mathbf{U\Phi U}^{-1} = \mathbf{U}\text{diag}(\boldsymbol{\varphi})\mathbf{U}^{-1}$. Let $\mathbf{T} := \mathbf{UV}^{-1}$. From condition (*ii*) we have that the Vandermonde matrix $\mathbf{\Psi}$ associated with $\mathbf{\Lambda}$ is non-singular and we can let $\mathbf{c} = \mathbf{\Psi}^{-1}\boldsymbol{\varphi}$. Then,

$$\begin{aligned}\mathbf{H}_s &= \mathbf{TV}h(\mathbf{\Lambda})\mathbf{V}^{-1}\mathbf{T}^{-1} = \mathbf{TV}\text{diag}(\mathbf{\Psi c})\mathbf{V}^{-1}\mathbf{T}^{-1} \\ &= \mathbf{U}\text{diag}(\boldsymbol{\varphi})\mathbf{U}^{-1} = \mathbf{B}.\end{aligned}$$

□

If conditions for Theorem 1 do not hold, for example if the degree $L-1$ of the filter is less than $N-1$ (where N is the dimension of the signal), it may not be possible to achieve perfect reconstruction. Therefore, we consider the case where we aim to find the filter parameters for an approximating filter $\tilde{\mathbf{H}}_s$ yielding the minimum approximation error. To arrive at an objective criterion in terms of the linear operator $\mathbf{B}$ and the parameters of the approximating filter $\tilde{\mathbf{H}}_s$ we start with the zero mean signal $\mathbf{x}$ and define the error signal $\mathbf{z} = \tilde{\mathbf{H}}_s\mathbf{x} - \mathbf{Bx}$. Expressing the covariance of the error signal as $\mathbf{R} := \mathbb{E}[\mathbf{zz}^T] = (\tilde{\mathbf{H}}_s - \mathbf{B})(\tilde{\mathbf{H}}_s - \mathbf{B})^T$ and taking its trace, we note that $\text{trace}\,(\mathbf{R}) = \sum_{i,j=1}^N |[\tilde{\mathbf{H}}_s - \mathbf{B}]_{ij}|^2 = \|\tilde{\mathbf{H}}_s - \mathbf{B}\|_{\mathcal{F}}^2$ and propose this objective as an approximation criterion. In addition, we would like to quantify the accuracy provided by the best achievable approximation in that criterion. These results are formalized in the next theorem.

Theorem 2. (Approximate reconstruction) *Let $\mathbf{B} \in \mathbb{C}^{N\times N}$ be a target linear operator with eigendecomposition given by $\mathbf{B} = \mathbf{U}diag(\boldsymbol{\varphi})\mathbf{U}^{-1}$, and let $\mathbf{S} \in \mathbb{C}^{N\times N}$ be the base shift, with eigendecomposition $\mathbf{S} = \mathbf{V\Lambda V}^{-1}$. In addition, let $\tilde{\mathbf{H}}_s$ as in* (1) *be the approximating filter of reduced order $L-1 < N-1$, with associated filter coefficients $\tilde{\mathbf{c}} \in \mathbb{C}^L$ and similarity transformation matrix $\mathbf{T} = \mathbf{UV}^{-1}$. Define the difference vector $\boldsymbol{\delta}_\varphi(\tilde{\mathbf{c}}) := \hat{\boldsymbol{\varphi}}(\tilde{\mathbf{c}}) - \boldsymbol{\varphi} \in \mathbb{C}^N$, where $diag(\hat{\boldsymbol{\varphi}}(\tilde{\mathbf{c}})) = diag(\mathbf{\Psi}\tilde{\mathbf{c}}) = \tilde{h}_s(\mathbf{\Lambda})$ is the approximation by $\tilde{\mathbf{H}}_s$ of $diag(\boldsymbol{\varphi})$, the frequency response of $\mathbf{B}$. Then, the error from the approximate implementation of the linear transformation $\mathbf{B}$ by the linear filter $\tilde{\mathbf{H}}_s$ in the Frobenius norm is $\|\mathbf{B} - \tilde{\mathbf{H}}_s\|_{\mathcal{F}}^2 = \|\boldsymbol{\delta}_\varphi(\tilde{\mathbf{c}})\|_2^2$.*

Furthermore, the optimal filter coefficients with respect to this error are a solution to $\tilde{\mathbf{c}}^* = \arg\min_{\tilde{\mathbf{c}}\in\mathbb{C}^L} \|\boldsymbol{\delta}_\varphi(\tilde{\mathbf{c}})\|_2^2$, whose closed-form solution is obtained as:

$$\tilde{\mathbf{c}}^* = \left(\mathbf{\Psi}^T\mathbf{\Psi}\right)^{-1}\mathbf{\Psi}\boldsymbol{\varphi} = \mathbf{\Psi}^\dagger\boldsymbol{\varphi}, \tag{4}$$

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and corresponding error given by

$$\min_{\tilde{\mathbf{c}}\in\mathbb{C}^L} \|\boldsymbol{\delta}_\varphi(\tilde{\mathbf{c}})\|_2^2 = \left\| \left(\mathbf{I} - \boldsymbol{\Psi}\boldsymbol{\Psi}^\dagger\right)\boldsymbol{\varphi} \right\|_2^2 . \tag{5}$$

◇

Proof. Let $\hat{\boldsymbol{\Phi}} := \text{diag}(\hat{\boldsymbol{\varphi}})$, $\boldsymbol{\Phi} := \text{diag}(\boldsymbol{\varphi})$, and $\boldsymbol{\Delta}_\Phi := \hat{\boldsymbol{\Phi}} - \boldsymbol{\Phi} = \text{diag}(\boldsymbol{\delta}_\varphi)$. Then

$$\begin{aligned}
\left\|\mathbf{B} - \tilde{\mathbf{H}}_s\right\|_{\mathcal{F}}^2 &= \left\| \mathbf{B} - \mathbf{T}\mathbf{V}\tilde{h}(\boldsymbol{\Lambda})\mathbf{V}^{-1}\mathbf{T}^{-1} \right\|_{\mathcal{F}}^2 \\
&= \left\| \mathbf{B} - \mathbf{T}\mathbf{V}\text{diag}(\boldsymbol{\Psi}\tilde{\mathbf{c}})\mathbf{V}^{-1}\mathbf{T}^{-1} \right\|_{\mathcal{F}}^2 \\
&= \left\| \mathbf{U}\,\boldsymbol{\Phi}\,\mathbf{U}^{-1} - \mathbf{T}\,\mathbf{V}\,(\boldsymbol{\Phi} + \boldsymbol{\Delta}_\Phi)\,\mathbf{V}^{-1}\mathbf{T}^{-1} \right\|_{\mathcal{F}}^2 \\
&= \left\| \mathbf{U}\,\boldsymbol{\Phi}\,\mathbf{U}^{-1} - \mathbf{U}\,(\boldsymbol{\Phi} + \boldsymbol{\Delta}_\Phi)\,\mathbf{U}^{-1} \right\|_{\mathcal{F}}^2 \\
&= \left\| \mathbf{U}\boldsymbol{\Delta}_\Phi\mathbf{U}^{-1} \right\|_{\mathcal{F}}^2 = \| \boldsymbol{\Delta}_\Phi \|_{\mathcal{F}}^2 = \|\boldsymbol{\delta}_\varphi\|_2^2 .
\end{aligned}$$

For the optimal coefficients $\tilde{\mathbf{c}}^*$, we note that

$$\arg\min_{\tilde{\mathbf{c}}\in\mathbb{C}^L} \|\boldsymbol{\delta}_\varphi(\tilde{\mathbf{c}})\|_2^2 = \arg\min_{\tilde{\mathbf{c}}\in\mathbb{C}^L} \|\boldsymbol{\varphi} - \boldsymbol{\Psi}\tilde{\mathbf{c}}\|_2^2 ,$$

whose least squares solution is given by finding the pseudo-inverse of the Vandermonde matrix $\boldsymbol{\Psi}$ and letting $\tilde{\mathbf{c}}^* = \boldsymbol{\Psi}^\dagger\boldsymbol{\varphi}$. Substituting the solution in the objective function gives the minimum error in (5). □

Discussion

Similarity Transformation and Graph Filters

The similarity transformation $\mathbf{T}$ can be interpreted, from the Fourier transform standpoint as follows. Given an input signal $\mathbf{x}$, the matrix $\mathbf{T}^{-1} = \mathbf{V}\mathbf{U}^{-1}$ performs a composition of a the Fourier transform $\mathbf{U}^{-1}$ induced by the target operator $\mathbf{B}$ and the inverse graph Fourier transform of the base shift $\mathbf{S}$. As explored in Theorems 1 and 2, the similarity transformation defined matches the Fourier bases of $\mathbf{S}$ and $\mathbf{B}$, and, in doing so, it reduces the filter design problem to the one of finding the coefficients $\mathbf{c}$ that approximate the filter frequency response $h(\mathbf{S})$ to the spectrum $\boldsymbol{\varphi}$ of the target linear operator $\mathbf{B}$. Additionally, if one wishes to interpret the similarity shift $\mathbf{A}$ in relation to graph filters, the graph $\mathcal{G}_\mathbf{A}$ associated with $\mathbf{A}$ represents a graph that is cospectral [10] to the graph $\mathcal{G}_\mathbf{S}$ associated with the base shift $\mathbf{S}$. It should be noted that the set of edges of $\mathcal{G}_\mathbf{A}$ will in general not coincide with the set of edges of $\mathcal{G}_\mathbf{S}$. The implementation of the similarity-shift will involve information from node neighborhoods defined in terms of $\mathcal{G}_\mathbf{A}$ instead of $\mathcal{G}_\mathbf{S}$.

A Local Implementation of Linear Filters

In some cases, it might be necessary to consider a distributed implementation of the filters, as it is proposed in [9]. Nevertheless, it requires a "design" stage which is centralized. In other words, there is a step in which the network information is given to a designer who solves a computation problem in order to determine the parameters that specify the filter. After this stage, the designer provides the set of parameters to the different nodes (or agents) in the network, which in turn, will implement a distributed protocol, i.e., each node exclusively uses information originated in their neighbors (alongside the information it received from the designer). Under the same hypotheses, i.e., the existence of a designer that detains knowledge of the general structure of the network, it is possible to obtain a distributed implementation of a filter by considering the following steps (see details, for instance, in [11]):

1. During the design stage, the designer considers the shift $\mathbf{S}$ and target operator $\mathbf{B}$. Given the neighborhood $\mathcal{N}_i = \left\{k_1^i, \ldots, k_{p_i}^i\right\}$ of each node, a local signal mapping matrix is defined as $\mathbf{C}_i := [\,\mathbf{e}_{k_1^i} | \ldots | \mathbf{e}_{k_{p_i}^i}]^T \in \mathbb{C}^{p_i\times N}$ and defines the local history matrix for $\bar{L}$ observations $\mathbf{G}_i := [(\mathbf{C}_i\mathbf{S}^0)^T | \ldots | (\mathbf{C}_i\mathbf{S}^{\bar{L}-1})^T]^T \in \mathbb{C}^{\bar{L}p_i\times N}$, which is used to determine the vector of local filter weights $\mathbf{w}_i := (\mathbf{b}_i\mathbf{G}_i^\dagger)^T \in \mathbb{C}^{\bar{L}p_i}$ supplied to each node, where $\mathbf{b}_i$ is the i-th row of $\mathbf{B}$.
2. At the implementation stage, each node i iteratively accesses the local signal $\mathbf{y}_i(k) = \mathbf{C}_i\,\mathbf{x}(k)$ and computes $\mathbf{z}_i(\bar{L}-1) = \sum_{k=0}^{\bar{L}-1} \left[\,[\mathbf{w}_i]_{p_ik+1}, \ldots, [\mathbf{w}_i]_{p_i(k+1)}\,\right]\mathbf{y}_i(k)$, which implies that at time $\bar{L}-1$ each node achieves $\mathbf{z}_i(\bar{L}-1) = \mathbf{b}_i\,\mathbf{x}(0)$ as the filter output.

Such solution is feasible when $\text{rank}(\mathbf{G}_i) = N$, and the number of observations $\bar{L}$ is at most N, this filter implements exactly the target linear operator $\mathbf{B}$. In practice, one finds that the requirement $L \geq N$ can be typically relaxed, with exact implementation being achieved for values of $\bar{L}$ considerably less than N [12, Ch. 6]. Therefore, the lowest order for the filter is given by $\min(L, \bar{L})$.

Given that a local implementation solution has been established, it remains to address if it is possible to achieve the whole process in a fully distributed setting, i.e., without the existence of a designer that holds the overall information of the network. In other words, each node should, by itself, find a protocol (filter design and implementation) which it should perform in a distributed way, i.e., using exclusively information from its neighbors.

IV. Numerical Experiments

In the next subsections, we apply the similarity shift method to two distinct cases. First, we define a random network model and vary the level of mismatch between the Fourier basis of the base shift and that of the target operator, which is derived as controlled perturbation of the base shift. Second, we apply the similarity shift method to a real dataset of structural and functional brain networks obtained from neuroimaging. The similarity shift method is compared with the *node-variant* and *node-invariant* graph filters presented in [9]. These filters are polynomials of the base shift operator of degree up to $L-1$, whose coefficients are allowed to be node-specific in the first case, and common to all nodes in the second case.

Random Networks

In this experiment, we follow the set up presented in [9]. More specifically, we generate $M = 1.000$ sample Erdős-Rényi random networks with edge probability 0.3, for two different graph sizes $N = 10$ and $N = 50$. Each random graphs is identified with a base shift $\mathbf{S}$, which is then scaled to that the maximum absolute value of its eigenvalues is 1. Since the graph is symmetric, it is assured to be diagonalizable, and we perform an eigendecomposition $\mathbf{S}_i = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^T$. One target linear operator is generated for each sample network by letting $\mathbf{B} = \mathbf{V}_\mathbf{B}^Q\text{diag}(\boldsymbol{\beta})(\mathbf{V}_\mathbf{B}^Q)^{-1}$, where the entries of $\boldsymbol{\beta}$ are generated from a standard normal distribu-

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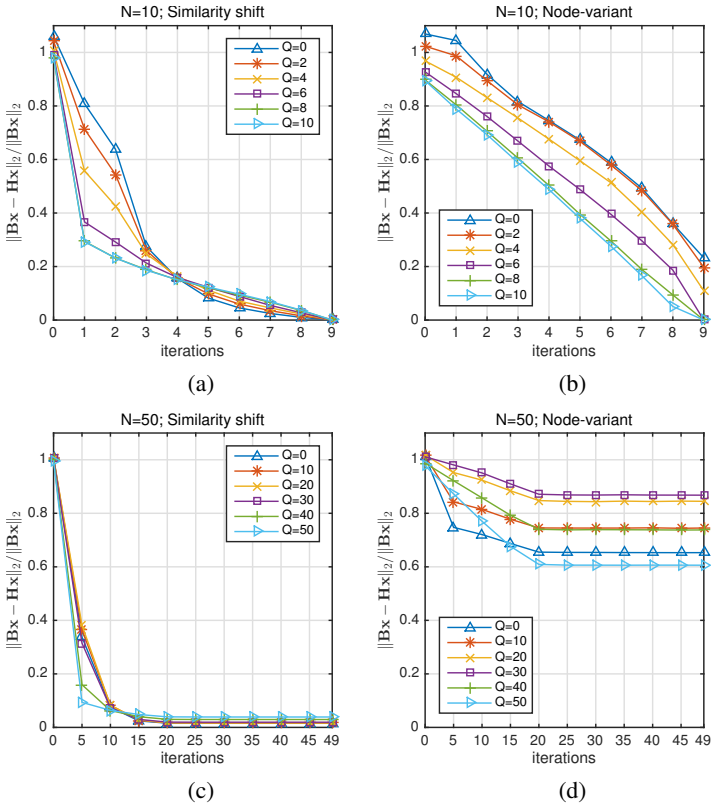


Fig. 1. Performance for random networks with perturbed eigenvectors for similarity shift and node-variant methods for $N = 10$ and $N = 50$.

tion. The eigenbasis of each sample $\mathbf{B}$ is chosen as follows. We choose, from the corresponding sample shift, Q columns from its eigenbasis $\mathbf{V_B}$ randomly and without replacement, while the remaining $N - Q$ columns are randomly chosen to form a basis for the complementary subspace orthogonal to the first Q columns. As the $N - Q$ columns are not constrained to be orthogonal to each other, the resulting matrix $\mathbf{B}$ does not need to be symmetric. We refer to Q as the number of *shared* eigenvectors, which is a parameter varied in the simulations. We define the approximation error for each sample as $e := \|\mathbf{Hx} - \mathbf{Bx}\|_2 / \|\mathbf{Bx}\|_2$ averaged over random input signals $\mathbf{x} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_N)$. We evaluate the error for varying number of shared eigenvectors and for increasing values of L, up to $N - 1$ in both $N = 10$ and $N = 50$. In Figure 1, we plot the evolution of performance of the sample with the median error across the $M = 1.000$ samples for the similarity-shift method, as well as for the node-variant method. Notice in (a) that for the similarity shift method, when $N = 10$ and $L > 3$, the median error is under 0.2 for all Q, while the node-variant method in (b) achieves comparable values only for $L = 9$. For the larger network size $N = 50$, the node-variant implementation in (d) presents an error above 0.5 for all Q and L. The error for the similarity-shift implementation in (c) is below 0.1 for all Q and $L > 10$. Therefore, we can argue that the similarity transformation compensates the effect of the perturbation in the mutual eigenbasis.

Brain Networks

The second experiment considers the problem of mapping two types of networks commonly studied in the field of network neuroscience, which we refer to structural and functional networks [13]. Structural networks represent the inter-regional connections established by white matter neuronal tracts in the brain. These are obtained from diffusion MRI measurements, followed by a tractography algorithm [14]. Functional networks are obtained by evaluating the time correlation (or frequency coherence) between the time series of the fMRI signal measured at different brain regions. In this experiment, we identify the structural matrix with the base shift $\mathbf{S}$, and the functional matrix with the target operator $\mathbf{B}$, for each of the $M = 84$ paired samples in the dataset [14, 15]. We define the approximation error between the filter and the linear operator as $e_{\mathcal{F}} := \|\mathbf{H} - \mathbf{B}\|_{\mathcal{F}} / \|\mathbf{B}\|_{\mathcal{F}}$ and evaluate it for different filter degrees, up to $N = 90$ the number of nodes in both graphs. In Figure 2, we present boxplots summarizing the distribution of the approximation error, over all samples, when we applied the similarity filter, as well as the node-variant and node-invariant methods. Notice that the node-invariant stabilizes its performance with median error over 0.6 for the maximum degree $L = N = 90$, and node-variant stabilizes with median error above 0.3 for $L = N = 90$. The similarity shift method was able to align the eigenbasis of the structural with the functional matrices, and match both frequency responses, achieving a very low median error, below 0.01, at $L = N = 90$. These results highlight the benefits of the proposed method.

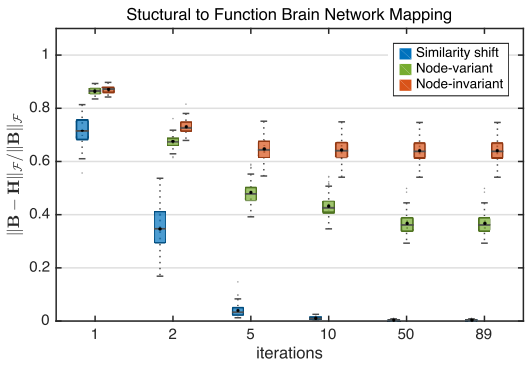


Fig. 2. Linear operator approximation for a real dataset of structural and functional brain networks with 84 individuals. The middle bar indicates the median error over the samples, while the dot indicates the mean.The bottom and top ends of each the solid bar indicate the first and third quartiles of the error over the samples.

V. Conclusion

In this paper we proposed a method to design linear filters based on generalized shift operator. The generalized shift operator is characterized by a similarity transformation matrix which has been explicitly identified. In addition, we determined conditions for perfect and optimal approximate implementations, quantifying the incurred error in the latter case. For the numerical experiments performed, it was observed that the similarity filters needed comparatively fewer iterations and were applicable to signals of larger dimensions than existing filters. Among future research directions, the authors would like to consider the problem of finding similarity shifts under sparsity constraints. Additionally, it would be interesting to address the possibility of a protocol that enabling the design of filters in a fully distributed manner.

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