Consistent hierarchies of nonlinear abstractions

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Abstract

In this paper, we consider the problem of constructing hierarchies of nonlinear control systems that preserve reachability properties, and, in particular, local accessibility. In this hierarchical framework, showing local accessibility of the higher level *abstracted* model of the nonlinear control system is *equivalent* to showing local accessibility of the, more detailed, lower level model. Hierachies of consistent nonlinear abstractions can therefore result in significant complexity reduction in determing the reachability properties of nonlinear systems.

1 Introduction

One of the main challenges in hierarchical systems is the extraction of a hierarchy of models at various levels of abstraction which are compatible with the functionality and objectives of each layer. In addition, the use of hierarchies is also useful in the analysis of complex systems. For example, in order to verify that a given large scale system satisfies certain properties, one tries to extract a simpler but qualitatively equivalent abstracted system. Checking the desired property on the abstracted system should be *equivalent* to checking the property on the original system. Depending on the property, special quotient systems which preserve the property of interest are constructed.

As a result, the notions of *abstraction* or *aggregation* refer to grouping the system states into equivalence classes. Depending on the cardinality of the resulting quotient space we may have *discrete* or *continuous* abstractions. With this notion of abstraction, the abstracted system is defined as the induced quotient dynamics. Hierarchical abstractions for *discrete* event systems have been formally considered in control community by [2, 9].

In previous work, we have focused on extracting *continuous abstractions* from continuous systems. In particular, in [8] a hierarchical framework for continuous control systems was conceptualized and formally proposed. In [6, 7], easily checkable characterizations were obtained for constructing reachability preserving abstractions of linear control systems. In this paper, we extend our hierarchical approach to a significant class of nonlinear control systems that consists of analytic control systems on analytic manifolds. In particular, we address the following problem.

Problem 1.1 *Given an* analytic *control system*

$$\dot{x} = f(x, u) \qquad x \in \mathbb{R}^n \;\; u \in \mathbb{R}^m \tag{1}$$

and an analytic *map* $y = \Phi(x)$, *where* $\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}^p$, $p \leq n$, *construct a control system*

$$\dot{y} = g(y, v) \qquad y \in \mathbb{R}^p \;\; v \in \mathbb{R}^k \tag{2}$$

which can produce as trajectories all functions of the form $y(t) = \Phi(x(t))$, *where* $x(t)$ *is a trajectory of system (1). Furthermore, characterize analytic maps* Φ *for which (1) is locally accessible (controllable) if and only if (2) is locally accessible (controllable).*

The function Φ is the *quotient map* which performs the state aggregation. System (2) will be referred to as the *abstraction* or *macromodel* of the finer *micromodel* (1). In [6, 7], we extended the geometric notion of Φ-related vector fields to control systems, which allowed us to push forward control systems through quotient maps and obtain well defined control systems describing the aggregate dynamics. Furthermore, in [6, 7], we were able to provide constructive formulas for easily generating linear abstractions of linear control systems.

In this paper, we provide a constructive method for extracting abstractions for analytic control systems on analytic manifolds. Our method is the natural nonlinear generalization of the linear method provided in [6, 7]. Furthermore, the method is natural in the sense that it constructs the *smallest* Φ-related or abstracted control system.

We then consider the problem of extracting abstractions while preserving the property of local accessibility [5]. We determine conditions on the map Φ under

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which local accessibility of the abstracted system (2) is equivalent to local accessibility of system (1). Such conditions ensure that the macromodel is a *consistent abstraction* of the micromodel in the sense that reachability requests from the macromodel are implementable by the micromodel. In addition, they greatly reduce the complexity of determining local accessibility properties of nonlinear control systems, since, rather than checking controllability of a large scale nonlinear system, we can construct a sequence of consistent abstractions and then check the local accessibility of systems which are much smaller in size.

The structure of this paper is as follows: In Section 2 we revisit some results from [6, 7, 8], whereas in Section 3 we define consistent abstractions. In Section 4 we restrict these notions to analytic control systems and provide methods for constructing analytic abstractions and characterize consistent abstractions. Section 5 applies our results to various classes of analytic systems. Due to space restrictions, all proofs are omitted.

2 Φ-Related Control Systems

We first review some definitions from differential geometry [1]. Let M be a differentiable manifold and T_pM be the tangent space of M at $p \in M$. We denote by $TM = \bigcup_{p\in M} T_pM$ the tangent bundle of M and by π the canonical projection map $\pi : TM \longrightarrow M$ taking a tangent vector $X_p \in T_pM \subset TM$ to the point $p \in M$. Now let M and N be smooth manifolds and $\Phi : M \longrightarrow N$ be a smooth map. Let $p \in M$ and let $q = \Phi(p) \in N$. We push forward tangent vectors from T_pM to T_qN using the induced push forward map $\Phi_* : T_pM \longrightarrow T_qN$, sometimes also denoted by $T\Phi$.

A vector field on a manifold M is a smooth map $X : M \longrightarrow TM$ which assigns to each point p of M a tangent vector in T_pM. Let $I \subseteq \mathbb{R}$ be an open interval containing the origin. An integral curve of a vector field is a smooth curve $c : I \longrightarrow M$ that satisfies $c' = c_*(1) = X \circ c(t)$ for all $t \in I$ where $c_*(1)$ denotes $c_*(\frac{d}{dt})$.

In this paper, an *abstraction* or *aggregation map* is a map $\Phi : M \longrightarrow N$ which we will assume to be a smooth (or analytic) submersion. Given a vector field X on manifold M and a smooth map $\Phi : M \longrightarrow N$, not necessarily a diffeomorphism, the push forward of X by Φ_* is generally not a well defined vector field on N. This leads to the concept of Φ-related vector fields.

Definition 2.1 *Let X and Y be vector fields on manifolds M and N respectively and $\Phi : M \longrightarrow N$ be a smooth map. Then X and Y are Φ-related iff $\Phi_* \circ X = Y \circ \Phi$.*

If Φ is not surjective then X may be Φ-related to many vector fields on N. The following well known theorem gives us a condition on the integral curves of two Φ-related vector fields.

Theorem 2.2 ([1]) *Let X and Y be vector fields on M and N respectively and let $\Phi : M \longrightarrow N$ be a smooth map. Then vector fields X and Y are Φ-related if and only if for every integral curve c of X, $\Phi \circ c$ is an integral curve of Y.*

Even though Φ-relatedness of vector fields is a rather restrictive condition, this is not the case for control systems. In [6, 7, 8], Definition 2.1 and Theorem 2.2 were extended to control systems. We briefly review some of the results of those papers. We first begin with a global and coordinate-free description of control systems.

Definition 2.3 ([5]) *A control system $S = (B, F)$ consists of a fiber bundle $\pi : B \longrightarrow M$ called the control bundle and a smooth map $F : B \longrightarrow TM$ which is fiber preserving, that is $\pi' \circ F = \pi$ where $\pi' : TM \longrightarrow M$ is the tangent bundle projection. Given a control system $S = (B, F)$, the control distribution $\mathcal{D}$ of control system S, is naturally defined pointwise by $\mathcal{D}(p) = F(\pi^{-1}(p))$ for all $p \in M$.*

Essentially, the base manifold M of the control bundle is the state space and the fibers $\pi^{-1}(p)$ can be thought of as the state dependent control spaces. Given the state p and the input, the map F selects a tangent vector from T_pM. The notion of trajectories of control systems is now defined.

Definition 2.4 *A smooth curve $c : I \longrightarrow M$ is called a trajectory of the control system $S = (B, F)$ if there exists a curve $c^B : I \longrightarrow B$ satisfying*

$$\pi \circ c^B = c$$
$$c' = c_*(1) = F \circ c^B$$

In local (bundle) coordinates, Definition 2.4 simply says that a trajectory of a control system is a curve $x\colon I \to M$ for which there exists a function $u\colon I \to U$ satisfying, $\dot{x} = F(x, u)$. Note that even though Definition 2.4 assumes c to be smooth, the bundle curve c^B is not necessarily smooth. The definition therefore allows nonsmooth control inputs as long as the projection $\pi \circ c^B = c$ is smooth. We can now define Φ-related control systems in a manner similar to Definition 2.1 for vector fields.

Definition 2.5 *Let $S_M = (B_M, F_M)$ with $\pi_M : B_M \longrightarrow M$ and $S_N = (B_N, F_N)$ with $\pi_N : B_N \longrightarrow N$ be two control systems. Let $\Phi : M \longrightarrow N$ be a smooth map. Then control systems S_M and S_N are Φ-related iff for every $p \in M$*

$$\Phi_* \circ F_M\left(\pi_M^{-1}(p)\right) \subseteq F_N\left(\pi_N^{-1}(\Phi(p))\right) \qquad (3)$$

In other words, if $\mathcal{D}_M$ and $\mathcal{D}_N$ are the control distributions associated with control systems S_M and S_N respectively, then S_M and S_N are Φ-related if at every $p \in M$ we have $\Phi_*(\mathcal{D}_M(p)) \subseteq \mathcal{D}_N(\Phi(p))$. Control system S_N will be referred to as an *abstraction* of control system S_M ([6, 7, 8]). Note that many control systems S_N may be Φ-related to S_M as the set of tangent vectors on N that must be captured, can be generated using many control parameterizations.

In contrast to the restrictive conditions of Theorem 2.2 for vector fields, the following proposition, shows that every control or dynamical system is Φ-related to some control system for any map Φ.

Proposition 2.6 *Given any control system $S_M = (B_M, F_M)$ and any smooth map $\Phi : M \longrightarrow N$, then there exists a control system $S_N = (B_N, F_N)$ which is Φ-related to S_M. In particular, every vector field X on M is Φ-related to some control system S_N.*

The following theorem should be thought of as a generalization of Theorem 2.2 for control systems.

Theorem 2.7 ([6, 7]) *Let $S_N = (B_N, F_N)$ and $S_M = (B_M, F_M)$ be two control systems and $\Phi : M \longrightarrow N$ be a smooth map. Then S_M and S_N are Φ-related if and only if for every trajectory c_M of S_M, $\Phi \circ c_M$ is a trajectory of S_N.*

If Σ_{S_M} and Σ_{S_N} denote all trajectories of control systems S_M and S_N respectively, then Theorem 2.7 simply states that S_M and S_N are Φ-related if and only if $\Phi(\Sigma_{S_M}) \subseteq \Sigma_{S_N}$. The quotient system therefore overapproximates the abstracted trajectories of the original system which may result in trajectories that the abstracted system S_N may generate but are infeasible in the original model S_M.

3 Consistent Control Abstractions

In addition to abstracting systems, we are also interested in propagating properties between the original and abstracted model. In [6, 7], we focused on controllability of linear control systems. In this paper, we focus on various notions of local accessibility for nonlinear systems.

Definition 3.1 *Let $S = (B, F)$ be a control system on M. For $p \in M$, define* $\mathrm{Reach}(p, S)$ *to be the set of points $q \in M$ for which there exists a trajectory $c : I \longrightarrow M$ of S such that for some $t_1, t_2 \in I$ we have $c(t_1) = p$ and $c(t_2) = q$.*

Definition 3.2 *A control system $S = (B, F)$ is*

- *Locally accessible at $p \in M$ if* $\mathrm{Reach}(p, S)$ *contains a non-empty, open set of M.*
- *Symmetrically locally accessible at $p \in M$ if* $\mathrm{Reach}(p, S)$ *contains a non-empty, open set of M which contains the point p.*
- *Locally accessible if it is locally accesible for every $p \in M$.*
- *Symmetrically locally accessible if it is symmetrically locally accesible for every $p \in M$.*
- *Controllable if for all $p \in M$,* $\mathrm{Reach}(p, S) = M$.

Definition 3.2 is slightly weaker from the standard definitions of local accessibility found in standard nonlinear control textbooks such as [3, 5]. In particular, Definition 3.2 does not constrain the trajectories of the control system to be within a prespecified neighborhood. Furthermore, we consider reachable states that can be reached in any time, rather than up to a certain bounded time interval. The fact that our notions are slightly weaker, however, allows us to maintain the following important results whose proofs are very similar to the ones found in as [3, 5].

Theorem 3.3 *Consider control system $S_M = (B_M, F_M)$ on a manifold M of dimension n, and the associated control distribution $\mathcal{D}_M$ defined for all $p \in M$ by $\mathcal{D}_M(p) = F_M(\pi_M^{-1}(p))$. Let $\mathcal{C}_M = \mathrm{Lie}(\mathcal{D}_M)$ be the accessibility Lie algebra generated by $\mathcal{D}_M$. Then*

- *If $\dim(\mathcal{C}_M(p)) = n$, then control system S_M is locally accessible at $p \in M$.*
- *If $\dim(\mathcal{C}_M(p)) = n$, and $\mathcal{D}_M$ is symmetric at p, then control system S_M is symmetrically locally accessible at $p \in M$.*
- *If $\dim(\mathcal{C}_M(p)) = n$ for all $p \in M$, then control system S_M is locally accessible.*
- *If $\dim(\mathcal{C}_M(p)) = n$ and $\mathcal{D}_M$ is symmetric for all $p \in M$, then control system S_M is symmetrically locally accessible.*
- *If $\dim(\mathcal{C}_M(p)) = n$ and $\mathcal{D}_M$ is symmetric for all $p \in M$, and M is a connected manifold, then S_M is controllable.*

Theorem 3.3 allows us to check the (symmetric) local accessibility of a control system by simply checking the rank of certain distributions. In this paper, however, we are interested in propagating local accessibility properties from a control system to its abstraction and vice versa. Theorem 2.7 allows us to easily propagate certain properties from the original control system to its abstractions.

Theorem 3.4 *Let control systems $S_M = (B_M, F_M)$ and $S_N = (B_N, F_N)$ be Φ-related with respect to some*

surjective submersion $\Phi : M \longrightarrow N$. Then for all $p \in M$,

$$\Phi\left(\mathrm{Reach}(p, S_M)\right) \subseteq \mathrm{Reach}(\Phi(p), S_N)$$

Thus, if S_M is (symmetrically) locally accessible (at $p \in M$) then S_N is also (symmetrically) locally accessible (at $\Phi(p) \in N$). Also, if S_M is controllable then S_N is controllable.

Note that Theorem 3.4 is true regardless of the structure of the map Φ as long as it is a smooth submersion. From a hierarchical perspective, the reverse question is a lot more interesting. In order to arrive at this goal, we define the notions of implementability and consistency.

Definition 3.5 *Let $S_M = (B_M, F_M)$ and $S_N = (B_N, F_N)$ be two control systems and $\Phi : M \longrightarrow N$ be a smooth submersion. Then S_N is* implementable *by S_M iff whenever there is a trajectory of S_N connecting $q_1 \in N$ and $q_2 \in N$, then there exist $p_1 \in \Phi^{-1}(q_1)$ and $p_2 \in \Phi^{-1}(q_2)$ and a trajectory of S_M connecting p_1 and p_2*

Implementability is therefore an existential property. If one thinks of the map Φ as a quotient map, then implementability requires that a reachability request is implementable by at least one member of the equivalence class.

Since implementability may depend on the particular element chosen from the equivalence class $\Phi^{-1}(q)$, it is resonable to make the controllability request well defined, by making the request independent of the particular element chosen from the equivalence class. This leads to the important notion of consistency.

Definition 3.6 *Let $S_M = (B_M, F_M)$ be a control system on M and $\Phi : M \longrightarrow N$ a smooth submersion. Then S_M is called* consistent *with respect to Φ whenever the following holds: if there exists a trajectory of S_M connecting p and q, then for all p' and for all q' such that $\Phi(p) = \Phi(p')$, $\Phi(q) = \Phi(q')$ there exists a trajectory connecting p' to q'.*

Implementability and consistency allow us to propagate the properties of (symmetric) local accessibility from the abstracted system S_N to the original system S_M.

Proposition 3.7 *Consider control systems $S_M = (B_M, F_M)$ and $S_N = (B_N, F_N)$ which are Φ-related with respect to smooth submersion $\Phi : M \longrightarrow N$. Assume that S_M is an implementation of S_N, and S_M is consistent. Then S_N is (symmetrically) locally accessible at $q \in N$ if and only if S_M is (symmetrically) locally accessible at every point $p \in \Phi^{-1}(q)$.*

In this section we identified the relevant notions for the study of controllability of Φ-related control systems. We also described them for arbitrary systems in terms of reachable sets. In the following sections, we will illustrate these notions, and give concrete characterizations of these concepts for large classes of nonlinear systems that result in analytic control distributions. Moreover, we show for such systems how to construct natural Φ-related systems with the desirable properties.

4 Consistent Abstractions of Analytic Systems

Let $S_M = (B_M, F_M)$ be a control system over a manifold M with control distribution $\mathcal{D}_M$ and let $\Phi : M \to N$ be a surjective submersion. From this point on, we assume that S_M, N and Φ are all analytic, and that N is an embedded submanifold of M.

Denote by K the distribution $\mathrm{Ker}(\Phi_*)$. Note that K is analytic and integrable; denote the foliation that integrates it by $\mathcal{K}$ and let $k = \dim K$. The basic idea in the construction of a control system S_N that is Φ-related to S_M is to first construct a distribution $\tilde{\mathcal{D}}_M$ on M which contains $\mathcal{D}_M$ and is K-invariant, that is, invariant with respect to vector fields in K. Then $\Phi_*(\tilde{\mathcal{D}}_M)$ can be taken to be the control distribution of S_N.

If $\mathcal{A}$ and $\mathcal{B}$ are two C^1 distributions on M, define a distribution $[\mathcal{A}, \mathcal{B}]$ by declaring $[\mathcal{A}, \mathcal{B}](p)$ to be the subspace of T_pM generated by vectors of the form $[X, Y](p)$, where X, Y are any two smooth vector fields in $\mathcal{A}$ and $\mathcal{B}$ respectively.

Definition 4.1 *Let $\tilde{\mathcal{D}}_M$ be the distribution on M generated by:*

$$K \cup \mathcal{D}_M \cup [K, \mathcal{D}_M] \cup [K, [K, \mathcal{D}_M]] \cup \cdots .$$

A graphical illustration of the construction of the distribution $\tilde{\mathcal{D}}_M$ is shown in Figure 1. Denote by $\tilde{S}_M$ any control system on M with control distribution $\tilde{\mathcal{D}}_M$.

Proposition 4.2 *(a) $\tilde{\mathcal{D}}_M$ is the smallest K-invariant distribution containing $\mathcal{D}_M$ and K.*

(b) For all $p_0, p_1 \in M$, if $\Phi(p_0) = \Phi(p_1)$, then

$$\Phi_* \tilde{\mathcal{D}}_M(p_0) = \Phi_* \tilde{\mathcal{D}}_M(p_1).$$

This allows us to make the following definition.

Definition 4.3 *Distribution $\mathcal{D}_N$ on N defined by*

$$\mathcal{D}_N(q) = \Phi_* \tilde{\mathcal{D}}_M(p),$$

for any $p \in \Phi^{-1}(q)$, is called canonically Φ-related to $\mathcal{D}_M$. *Any control system S_N with control distribution $\mathcal{D}_N$ is said to be* canonically Φ-related to S_M.

By Proposition 4.2, $\mathcal{D}_N$ is well defined. Furthermore:

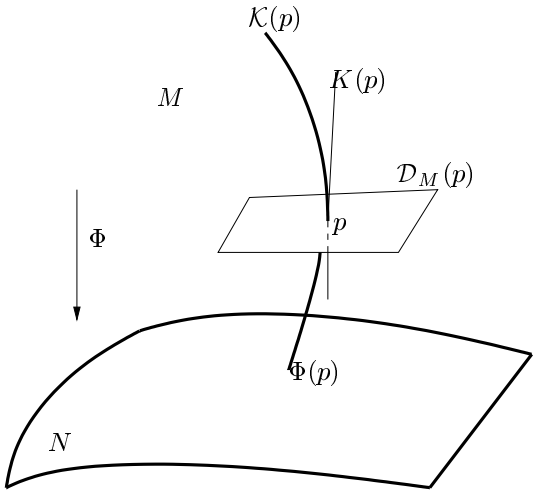


Figure 1: Construction of $\tilde{\mathcal{D}}_M$.

Theorem 4.4 *$\mathcal{D}_N$ is the smallest distribution that is Φ-related to $\mathcal{D}_M$.*

Definition 4.3 and Theorem 4.4 provide us with a constructive method to generate Φ-related systems. Furthermore, the construction is natural since it generates the smallest such system. We now study how the accessibility properties of S_N are related to S_M.

Proposition 4.5 *Assume $K \subset \text{Lie}(\mathcal{D}_M)$. Then:*

(a) $\text{Lie}(\tilde{\mathcal{D}}_M) = \text{Lie}(\mathcal{D}_M)$.

(b) For all $p \in M$: $\text{Reach}(p, \tilde{S}_M) = \text{Reach}(p, S_M)$.

(c) For all $p \in N$: $\mathcal{D}_N(p) = \tilde{\mathcal{D}}_M(p) \cap T_pN$.

(d) For all $p \in N$: $\text{Reach}(p, S_N) = \text{Reach}(p, \tilde{S}_M) \cap N$.

(e) For all $p \in N$: $\text{Reach}(p, S_N) = \text{Reach}(p, S_M) \cap N$.

The following theorem is an immediate consequence of the preceding result.

Theorem 4.6 *Let S_N be canonically Φ-related to S_M and assume (with the notation as above)*

$$\text{Ker}(\Phi_*) \subset \text{Lie}(\mathcal{D}_M). \tag{4}$$

Then:

(a) S_N is (symmetrically) locally accessible at $q \in N$ if and only if S_M is (symmetrically) locally accessible at every point $p \in \Phi^{-1}(q)$. Thus S_N is (symmetrically) locally accessible if and only if S_M is.

(b) S_N is controllable if and only if S_M is.

Therefore, as a consequence of the main Theorem 4.6, if S_M is Φ-related to S_N using the canonical construction described in Definition 4.3 and Theorem 4.4, and, condition (4) is satisfied, then S_M implements S_N, and, in addition, S_M is consistent. Therefore, for our canonical construction only the main condition (4) needs to be satisfied. In the next section, we apply Theorem 4.6 to various classes of analytic control systems.

5 Classes of Analytic Systems

In this section, we present a few interesting classes of analytic systems that are captured by Theorem 4.6. The first class recovers the results for linear systems that were presented in [6, 7].

5.1 Linear Systems

Consider the linear system

$$\dot{x} = Ax + Bu = Ax + \sum_{i=1}^{m} b_i u_i \tag{5}$$

where $x \in \mathbb{R}^n$ and b_i are constant input vector fields. Suppose we constrain our abstraction map to surjective linear maps $y = Cx$. For any such map, the construction of Theorem 4.4 results in the distribution generated at $y = Cx$ by

$$C\text{Ker}(C) \bigcup CAx \bigcup CB \bigcup C[\text{Ker}(C), Ax]$$

for any choice $x \in C^{-1}(y)$. Since C is surjective, we can choose $x = C^+y$ where $C^+ = C^T(CC^T)^{-1}$ is the pseudoinverse of C. Furthermore, $C\text{Ker}(C)$ is clearly zero, and higher order Lie brackets are also zero. Therefore the canonically C-related system for any linear surjective map $y = Cx$ is

$$\dot{y} = Fy + Gv \tag{6}$$

where

$$\begin{aligned} F &= CAC^+ \\ G &= [CB \;\; CAv_1 \;\; \ldots \;\; CAv_r] \end{aligned}$$

and $v_1, \ldots, v_r$ spanning $\text{Ker}(C)$. In order to propagate accessibility properties, the linear abstraction map must satisfy the consistency condition (4) which is $\text{Ker}(C) \subseteq \text{Lie}(Ax, b_1, \ldots, b_m) = Im[B \;\; AB \;\; \ldots \;\; A^{n-1}B] + \text{span}\{Ax\}$ which can always be satisfied as long as $B \neq 0$ since we can always choose $\text{Ker}(C) \subseteq Im[B]$. In order words, we can always obtain consistent abstractions as long as there are control inputs. For any such consistent constructions, local accessibility of (5) is *equivalent* to local accessibility of (5). For linear systems, however, the consistency condition propagates not only local accessibility but also controllability [6, 7]

5.2 Nonlinear systems with linear dynamics

Consider the following class of systems

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ \dot{u} &= Au + Bv \end{aligned} \tag{7}$$

where $x \in \mathbb{R}^{n_x}$, $u \in \mathbb{R}^{n_u}$, $v \in \mathbb{R}^{n_v}$, and f, g are analytic maps. Such systems frequently arise in mechanical systems with nonlinear kinematics but linear actuator dynamics. In studying the local accessibiliy of the above systems, rather than computing the full scale accessibility Lie algebra, one would like to decompose the analysis in order to reduce the complexity. For the simple projection map $\Phi(x,u) = x$, then the canonical constuction of Theorem 4.4 simply results in the Φ-related system

$$\dot{x} = f(x) + g(x)u \tag{8}$$

where $u \in \mathbb{R}^{n_u}$ is now an input. Local accessibility of (8) is equivalent to the local accessibility of (7) if the consistency condition (4) is satisfied. A little algebra reveals that the consistency condition is

$$\mathrm{Ker}(\Phi_*) = \mathrm{span}\{\begin{bmatrix} 0 \\ I \end{bmatrix}\} \subseteq \mathrm{span}\{\begin{bmatrix} \text{terms} \\ B \ \ldots A^{n_u}B \end{bmatrix}\} \tag{9}$$

In other words, if the pair (A,B) is controllable, then we can simply ignore the linear part of the system, and local accessibility of (8) is equivalent to the local accessibility of (7).

5.3 Strict feedback Systems

Consider the so-called strict feedback systems used in backstepping designs (see [4])

$$\begin{aligned} \dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 &= f_2(x_1,x_2) + g_2(x_1,x_2)u \end{aligned} \tag{10}$$

where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, and the maps f_1,f_2,g_1,g_2 are all analytic. Furthermore, suppose our abstraction map is the projection $\Phi(x_1,x_2) = x_1$. The canonical construction can be easily checked to result in the following abstracted system

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)x_2 \tag{11}$$

where x_2 is now an input. Notice that in this case, the abstraction of the original system which was affine in the controls is also affine. If $g_2(x_1,x_2)$ is full rank for all x_1,x_2, then the consistency condition is trivially satisfied and the the local accessibility of (10) is equivalent to the local accessibility of (11). If $g_2(x_1,x_2)$ is not full rank for all x_1,x_2, then the consistency condition may be satisfied using higher order Lie brackets.

6 Conclusions

In this paper,we provided constructive methods for abstracting analytic control systems with respect to analytic abstraction maps. Furthermore, we characterized abstraction maps that result in consistently propagating the property of local accessibility from one layer of the hierarchy to another. Our framework was then applied to various classes of analytic control systems.

There are many directions for further future research. The results of Sections 4 and 5 will provide hierarchical controllability algorithms for classes of nonlinear systems. Obtaining consistent abstractions for nonlinear systems with respect to stabilizability properties could be useful in classifying backsteppable systems. Other properties of interest include trajectory tracking, optimality and the proper propagation of state and input constraints.

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