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Bisimilar linear systems☆

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Abstract

The notion of bisimulation in theoretical computer science is one of the main complexity reduction methods for the analysis and synthesis of labeled transition systems. Bisimulations are special quotients of the state space that preserve many important properties expressible in temporal logics, and, in particular, reachability. In this paper, the framework of bisimilar transition systems is applied to various transition systems that are generated by linear control systems. Given a discrete-time or continuous-time linear system, and a finite observation map, we characterize linear quotient maps that result in quotient transition systems that are bisimilar to the original system. Interestingly, the characterizations for discrete-time systems are more restrictive than for continuous-time systems, due to the existence of an atomic time step. We show that computing the coarsest bisimulation, which results in maximum complexity reduction, corresponds to computing the maximal controlled or reachability invariant subspace inside the kernel of the observations map. These results establish strong connections between complexity reduction concepts in control theory and computer science.

Keywords: Bisimulation; Linear systems; Reachability; Transition systems; Control invariance

1. Introduction

Theoretical computer science, and, in particular, the areas of concurrency theory (Milner,1989), and computer-aided verification (Clarke, Grumberg, and Peled, 2001) have established formal notions of abstraction and model refinement which are used to tackle the state explosion arising in purely discrete systems. Given a discrete system, an *abstraction* is a quotient system that preserves some properties of interest while ignoring detail. Properties of interest include reachability, safety, liveness, and other properties expressible in various temporal logics (Pnueli 1977).

The notion of bisimulation (Milner, 1989) is one such formal notion of abstraction that has been used for reducing the complexity of finite state systems such as labeled transition systems. Bisimulations are partitions of the state space that preserve observations and reachability properties. In addition to reachability, bisimulations of finite transition systems preserve all properties that are expressible in temporal logics such as linear temporal logic (LTL), computational tree logic (CTL), and μ-calculus (Davoren & Nerode, 2000; Emerson, 1990). The notion of bisimulation has been also instrumental in obtaining decidability results for various classes of hybrid systems, by considering finite bisimulations of hybrid systems (see survey in Alur, Henzinger, Lafferriere, & Pappas, 2000). In the control community, notions that are similar to bisimulation have been considered in the hierarchical, supervisory control of discrete event systems (Caines & Wei, 1995; Wong & Wonham, 1995), and hybrid systems (see survey in Koutsoukos, Antsaklis, Stiver, & Lemmon, 2000). Furthermore, bisimulations have also been used as a controller synthesis tool for discrete-event systems (Barrett & Lafortune, 1998).

As mentioned in van der Schaft and Schumacher (2001), notions similar to bisimulation have escaped the world of purely continuous systems. Recently, a notion of abstraction, that is essentially the notion of *simulation* (Milner, 1989), was introduced for continuous-time systems in Pappas, Lafferriere, and Sastry (2000). In Pappas et al. (2000), a

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formal construction was provided for extracting abstractions of linear systems, and have, furthermore, characterized linear quotient maps that preserve control theoretic properties such as controllability (Pappas et al., 2000), and stabilizability (Pappas & Lafferriere, 2001). The complexity reduction properties of this approach to system analysis and design have been validated by recovering the best-known algorithms for controllability (Van Dooren, 1981), and stabilizability (Saad, 1988).

In this paper, we consider bisimilar linear systems in the exact sense that the notion is used in theoretical computer science. More precisely, given either a discrete- or continuous-time linear control system, and a finite (but affine) observation map, we consider a variety of labeled transition systems that are generated by the linear system and the observations. In particular, we consider *timed* and *time-abstract* transition systems T generated by linear systems, depending on whether we wish to maintain or ignore timing information on the labels of the transitions. Once the transition systems have been defined, we partition the state space using linear quotient maps of the form $z = Hx$, where two states are equivalent $x_1 \sim x_2$ if $Hx_1 = Hx_2$. Given such a partition $\sim$ of the state space, we construct a linear control system that generates the transitions of the quotient transition system $T/_\sim$. For both discrete- and continuous-time systems, the linear system that produces the transitions of $T/_\sim$ is H-related to the original linear system that generates the transitions of T. The notion of H-related control systems was introduced in Pappas et al. (2000) for continuous-time control systems.

We then characterize quotient maps $z = Hx$ that result in T and $T/_\sim$ being bisimilar. We first characterize quotient maps that preserve the observations. Then the crucial reachability preserving property of bisimulation, is characterized for all transition systems generated by discrete- and continuous-time linear systems. Since the property depends on the set of labels of the transition system, bisimulations of timed-transition systems require finer partitions than bisimulations of time-abstract transition systems which completely ignore timing information. Interestingly enough, bisimulations of discrete- and continuous-time systems have different characterizations when considering the timed transition systems they generate. This is due to the existence of an atomic time unit for discrete-time systems.

Timed bisimulations of discrete-time systems require the kernel of the quotient map be a controlled invariant subspace, and live inside the kernel of the observation map. Therefore, computing the *coarsest* bisimulation, which results in maximum complexity reduction, requires computing the largest controlled invariant subspace that lives inside the kernel of the observation map. Therefore, the well-known bisimulation algorithm for labeled transition systems takes the form of the standard maximal controlled invariant computation in control theory (Wonham, 1985). The characterizations for bisimilar continuous-time systems require a weaker condition than their discrete-time relatives, namely that the kernel of the quotient map be *reachability invariant*. The connection between control theoretic and computer science concepts is further amplified by recovering the exact model reduction results from control theory (Aoki, 1968) as a special case. This brings closer model minimization ideas from theoretical computer science and control theory.

The paper begins by reviewing labeled transition systems and bisimulations in Sections 2 and 3. In Section 4, we define various transition systems that are generated by discrete- and continuous-time linear systems. In Sections 5 and 6, the framework developed in Sections 2 and 3 is applied to the transition systems defined in Section 4, and we provide characterizations of bisimilar transition systems generated by linear systems. Section 7 translates the fixed point characterization of bisimulation into control theoretic subspace computations. Finally, in Section 8, we illustrate computations with an example.

2. Transition systems

Labeled transition systems can be thought of as graphs, possibly with an infinite number of states or transitions. In this paper, we consider labeled transition systems with deterministic observations, where each state of the transition system is mapped to a unique observation.

Definition 1 (Labeled transition systems). A labeled transition system with observations is a tuple $T = (Q, \Sigma, \rightarrow, O, \langle\langle\cdot\rangle\rangle)$ that consists of:

- a (possibly infinite) set Q of states,
- a (possibly infinite) set Σ of labels,
- a transition relation $\rightarrow \subseteq Q \times \Sigma \times Q$,
- a (possibly infinite) set O of observations, and
- an observation map $\langle\langle\cdot\rangle\rangle : Q \rightarrow O$.

The transition $(q_1, \sigma, q_2) \in \rightarrow$ is commonly denoted as $q_1 \xrightarrow{\sigma} q_2$. The transition system is called *finite* if Q, Σ, and O are finite, and *infinite* otherwise. Every state q is mapped to a single observation $\langle\langle q\rangle\rangle \in O$. We will assume that the observation map $\langle\langle\cdot\rangle\rangle$ is surjective. A *region* is a subset $P \subseteq Q$ of the states. The σ-*successor* of a region P is defined as the set that can be reached from P with one σ-transition. More precisely, we define the following operator:

$$Post_\sigma(P) = \{q \in Q \mid \exists p \in P \text{ with } p \xrightarrow{\sigma} q\}. \tag{1}$$

The set of states that are accessible from P in two σ-transitions is $Post_\sigma(Post_\sigma(P))$, and is denoted $Post_\sigma^2(P)$. In general, $Post_\sigma^i(P)$ consists of the states that are accessible from P using i transitions with label σ. Similarly, $Post_\sigma^*(P) = \bigcup_{i \in \mathbb{N}} Post_\sigma^i(P)$ is the set of states that are *forward reachable* from P in any number of σ transitions.

A problem that is of great interest is the *reachability problem* which asks whether $Post_\sigma^*(Q_O) \cap Q_F \neq \emptyset$. If Q_F

represents an unsafe region of the state space, then solving the reachability problem corresponds to verifying whether the system is *safe*. In addition to safety properties, desired system specification may require more detailed system properties such as liveness, and fairness. Standard temporal logics such as linear temporal logic (LTL), computation tree logic (CTL), CTL*, and μ-calculus are used to formally specify such properties of systems (Pnueli, 1977). There is a very strong connection between bisimulation and temporal logics. In particular, bisimilar transition systems satisfy the same properties expressible in LTL, CTL*, and μ-calculus. In this paper, we focus on characterizing bisimilar linear systems, and refer the reader to a very rich literature on the interplay between bisimulations and temporal logic (Davoren & Nerode, 2000; Emerson, 1990).

3. Bisimilar transition systems

Property preserving partitions of the state space reduce the size of the system by ignoring modeling detail that is irrelevant to the properties of interest. Thus, given a transition system T, and partitions of the state space, we would like to first consider *quotient transition systems*.

Given transition system $T=(Q,\Sigma,\rightarrow,O,\langle\langle\cdot\rangle\rangle)$, and an equivalence relation $\sim$, the definition of *quotient transition system* $T/_\sim=(Q/_\sim,\Sigma,\rightarrow_\sim,O,\langle\langle\cdot\rangle\rangle_\sim)$ is natural. Let $Q/_\sim$ denote the quotient set, that is, the set of equivalence classes, and let $h:Q\rightarrow Q/_\sim$ be the quotient map. The set of labels Σ as well as the set of observations O of $T/_\sim$ are inherited from T. The transition relation $\rightarrow_\sim$ of $T/_\sim$ is induced from the transition relation of T. Therefore, if there exists a transition $q_1\xrightarrow{\sigma}q_2$ for T, then there exists a transition $h(q_1)\xrightarrow{\sigma}_\sim h(q_2)$ for $T/_\sim$. To complete the definition of $T/_\sim$, we define the observation map $\langle\langle\cdot\rangle\rangle_\sim:Q/_\sim\rightarrow O$ which maps any $h(q)\in Q/_\sim$ to $\langle\langle h(q)\rangle\rangle_\sim=o$ iff $\langle\langle q\rangle\rangle=o\in O$. In order for $\langle\langle\cdot\rangle\rangle_\sim$ to be well defined, we ask that the partition induced by $\sim$ be *observation preserving*, that is if $p\sim q$ then $\langle\langle p\rangle\rangle=\langle\langle q\rangle\rangle$. Therefore, equivalent states have the same observations. Even though many partitions may be observation preserving, a very natural observation preserving partition is *observational equivalence* where two states are *defined* to be equivalent if they are mapped to the same observation, that is $p\sim q$ if and only if $\langle\langle p\rangle\rangle=\langle\langle q\rangle\rangle$.

Observational equivalence partitions the state space of the transition system based on static observations. A partition of the state space that pays attention not only to static observations but also to dynamic sequences of observations is bisimulation.

Definition 2 (Bisimulations, Milner, 1989). Let $T=(Q,\Sigma,\rightarrow,O,\langle\langle\cdot\rangle\rangle)$ be a transition system. An observation-preserving equivalence relation $\sim$ is a bisimulation of T if for all states $p,q\in Q$ and for all labels $\sigma\in\Sigma$, the following property holds:

- if $p\sim q$ and $p\xrightarrow{\sigma}p'$, then there exists $q'\in Q$ such that $q\xrightarrow{\sigma}q'$ and $p'\sim q'$.

If $\sim$ is a bisimulation, then the quotient transition system $T/_\sim$ is called a *bisimulation quotient* of T, and the transition systems T and $T/_\sim$ are called *bisimilar*.[1] The crucial property of bisimulations states that equivalent states must be able to transition using the *same* label to states that are also equivalent. The following proposition will be a useful characterization of bisimulation for our goal of characterizing bisimilar linear systems.

Proposition 3 (Characterization). *Consider transition system T, and observation-preserving partition $\sim$ with quotient map $h:Q\rightarrow Q/_\sim$. Then $\sim$ is a bisimulation of T if and only if for all states $q\in Q$ and for all $\sigma\in\Sigma$, we have*

$$h(Post_\sigma(h^{-1}(h(q))))=h(Post_\sigma(q)). \tag{2}$$

Before proceeding with the straightforward proof, note that $h^{-1}(h(q))$ is simply the set of all states in Q that are equivalent to q.

Proof of Proposition 3. ($\Rightarrow$) Clearly, $h(Post_\sigma(q))\subseteq h(Post_\sigma(h^{-1}(h(q))))$ for any $q\in Q$. Assume now that $\sim$ is a bisimulation and let $p'=h(q')$ with $q'\in Post_\sigma(h^{-1}(h(q)))$. Therefore, there exists some $q_0\in h^{-1}(h(q))$ (therefore $q_0\sim q$) with $q_0\xrightarrow{\sigma}q'$. Since $\sim$ is a bisimulation, and $q_0\sim q$, there exists q'' such that $q\xrightarrow{\sigma}q''$ and $q'\sim q''$ thus $h(q')=h(q'')=p'$. Since $q''\in Post_\sigma(q)$ we have $p'\in h(Post_\sigma(q))$ and thus $h(Post_\sigma(q))\supseteq h(Post_\sigma(h^{-1}(h(q))))$.

($\Leftarrow$) Assume that $h(Post_\sigma(q))=h(Post_\sigma(h^{-1}(h(q))))$, and consider $q\xrightarrow{\sigma}q'$ and $p\sim q$. Thus $h(q)=h(p)$ and $q'\in Post_\sigma(q)$. Then $h(q')\in h(Post_\sigma(q))=h(Post_\sigma(h^{-1}(h(q))))=h(Post_\sigma(h^{-1}(h(p))))=h(Post_\sigma(p))$ since $p\sim q$. Thus, there exists $p'\in Post_\sigma(p)$ with $h(q')=h(p')$ and therefore $p\xrightarrow{\sigma}p'$ with $q'\sim p'$. □

Bisimilar systems have *equivalent* reachability properties, and in addition, bisimulations preserve all properties expressible in temporal logics such as LTL, CTL, CTL*, and even μ-calculus (Browne, Clarke, & Grumberg, 1988; Davoren & Nerode, 2000). Bisimulations are therefore used for complexity reduction since checking any property expressible by a temporal logic formula for T, can be performed equivalently on the bisimilar system $T/_\sim$, which is smaller in size.

[1] Bisimulations are also defined as symmetric simulation relations between two transition systems where one system simulates the transitions of the another while preserving the observations. In our context, where we work with equivalence relations, the quotient system $T/_\sim$ automatically simulates T by construction. The bisimulation property then ensures that T also simulates $T/_\sim$, hence T and $T/_\sim$ are bisimilar.

4. Linear systems as transition systems

The purpose of this paper is to characterize bisimilar transition systems that are *generated* by discrete- and continuous-time linear control systems. We begin with discrete-time linear systems whose dynamics are closer to transition systems due to the existence of an atomic time step.

4.1. Discrete-time linear systems

Consider a discrete-time linear system of the form

$$\Delta: \quad x_{k+1} = Ax_k + Bu_k \tag{3}$$

with time $k \in \mathbb{N}_+$, state $x_k \in \mathbb{R}^n$, control $u_k \in \mathbb{R}^m$, and matrices A, B of appropriate dimension. From linear systems theory, we know that given an initial condition x_0 at time zero, and an input sequence $\{u_i\}_{i=0}^{k-1} = \{u_0, u_1, \ldots, u_{k-1}\}$, then the state x_k at time k is given as

$$x_k = \Phi_\Delta(k, x_0, \{u_i\}_{i=0}^{k-1}) = A^k x_0 + \sum_{i=0}^{k-1} A^{k-i-1} Bu_i. \tag{4}$$

In addition to the discrete-time linear system, we must also provide a set of observations O. In linear system theory, one typically considers linear observations associated with a *surjective*, linear, output map $y = Cx$. In this paper, we shall consider a *finite* set of observations O associated with a finite set of affine predicates of the form $a_i x + b_i \leqslant 0$ where $0 \leqslant i \leqslant p$, $a_i \in \mathbb{R}^{1\times n}$, and $b_i \in \mathbb{R}$. Each predicate can be thought of as (discrete) observation map $\langle\langle\cdot\rangle\rangle_i : \mathbb{R}^n \to \mathbb{B} = \{0, 1\}$ defined as

$$\langle\langle x\rangle\rangle_i = \begin{cases} 1 & \text{if } a_i x + b_i \leqslant 0, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Collecting all p predicates results in an observation map $\langle\langle\cdot\rangle\rangle : \mathbb{R}^n \to \mathbb{B}^p$ defined as

$$\langle\langle x\rangle\rangle = \left(\langle\langle x\rangle\rangle_1, \langle\langle x\rangle\rangle_2, \ldots, \langle\langle x\rangle\rangle_p\right). \tag{6}$$

The discrete-time linear system Δ and the finite observation set $O = \mathbb{B}^p$ generate a variety of labeled transition systems. We consider three different labeled transition systems that differ in the amount of timing information that is retained or abstracted away on the transitions.

Definition 4 (One-step transition system T_Δ^1). Consider the discrete-time system Δ given by (3) and observation set $O = \mathbb{B}^p$. The one-step transition system $T_\Delta^1 = (Q, \Sigma, \to, O, \langle\langle\cdot\rangle\rangle)$ generated by Δ and O consists of:

- state space $Q = \mathbb{R}^n$,
- unique label $\Sigma = \{1\}$,
- transition relation $\to \subseteq Q \times \Sigma \times Q$ defined as

$$x \xrightarrow{1} x' \Leftrightarrow \exists u \text{ with } x' = \Phi_\Delta(1, x, u) = Ax + Bu, \tag{7}$$

- observations $O = \mathbb{B}^p$,
- observation map $\langle\langle\cdot\rangle\rangle : Q \to O$ given by (5) and (6).

The transitions of the one-step transition system naturally correspond to the evolution of the discrete-time system in one time step (hence the unique label 1 on the transitions). Furthermore, the transitions of Definition 4 are *control abstract* in the sense that the transition system does not care which particular control u is responsible for the transition of the discrete-time system. Therefore, even though the one-step transition system does maintain the timing information that is needed for a transition, it does not care *how* the transition is done since it ignores the control input that was used for the transition.

There are two natural variations of Definition 4. The first variation maintains not only one-step transitions, but also k-step transitions for any $k \in \mathbb{N}_+$ whereas the second variation does not care how many time steps are needed for a transition.

Definition 5 (Timed transition system $T_\Delta^{\mathbb{N}_+}$). Consider the discrete time system Δ given by (3) and observation set $O = \mathbb{B}^p$. The timed transition system $T_\Delta^{\mathbb{N}_+} = (Q, \Sigma, \to, O, \langle\langle\cdot\rangle\rangle)$ generated by Δ and O consists of:

- state space $Q = \mathbb{R}^n$,
- label set $\Sigma = \mathbb{N}_+$,
- transition relation $\to \subseteq Q \times \mathbb{N}_+ \times Q$ defined as

$$x \xrightarrow{k} x' \Leftrightarrow \exists \{u_i\}_{i=0}^{k-1} \quad \text{with } x' = \Phi_\Delta(k, x, \{u_i\}_{i=0}^{k-1}) \tag{8}$$

- observations $O = \mathbb{B}^p$,
- observation map $\langle\langle\cdot\rangle\rangle : Q \to O$ given by (5) and (6).

More intuitively, there exists a transition $x \xrightarrow{k} x'$ if there is an appropriate sequence of control inputs $\{u_i\}_{i=0}^{k-1}$ that in *exactly* k time steps will result in the discrete-time system Δ reaching state x' from state x. Since k-step transitions also include one-step transitions, it is clear that the transition relation of T_Δ^1 is strictly contained (as a set) in the transition relation of $T_\Delta^{\mathbb{N}_+}$.

Contrary to $T_\Delta^{\mathbb{N}_+}$ which maintains all timing information, the following transition system generated by discrete-time system Δ does not care about the exact number of time steps needed to reach a state. It simply maintains whether one state is reachable from another in any number of time steps. Since it abstracts away such timing information, it is called a time-abstract transition system.

Definition 6 (Time-abstract transition system T_Δ^τ). Consider the discrete time system Δ given by (3) and observation set $O = \mathbb{B}^p$. The time-abstract transition system $T_\Delta^\tau = (Q, \Sigma, \to, O, \langle\langle\cdot\rangle\rangle)$ generated by Δ and O consists of:

- state space $Q = \mathbb{R}^n$,
- unique label $\Sigma = \{\tau\}$,

- transition relation $\to \subseteq Q \times \{\tau\} \times Q$ defined as

$$x \xrightarrow{\tau} x' \Leftrightarrow \exists k \in \mathbb{N}_+ \exists \{u_i\}_{i=0}^{k-1} \quad \text{with } x' = \Phi_\Delta(k, x, \{u_i\}_{i=0}^{k-1}), \tag{9}$$

- observations $O = \mathbb{B}^p$,
- observation map $\langle\langle\cdot\rangle\rangle : Q \to O$ given by (5) and (6).

In other words, a transition $x \xrightarrow{\tau} x'$ occurs if x' is reachable from x in any number of steps by an appropriate sequence of control inputs. Therefore, T_Δ^τ is both time- and control-abstract and simply preserves reachability properties.

4.2. Continuous-time linear systems

The transition systems generated by continuous-time systems are conceptually similar to the transition systems generated by discrete-time systems. Consider a continuous-time linear system

$$C: \dot{x} = Ax + Bu \tag{10}$$

with time $t \in \mathbb{R}_+$, state $x(t) \in \mathbb{R}^n$, control $u(t) \in \mathbb{R}^m$, and matrices A, B of appropriate dimension. Given an initial condition x_0, and an input function $u_{[0,t]}$ defined on interval $[0, t]$, the explicit solution or *flow* of the linear differential equation (10) is

$$x(t) = \Phi_C(t, x_0, u_{[0,t]}) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau. \tag{11}$$

Whereas Definitions 5 and 6 for discrete-time systems have natural continuous-time counterparts, there is no natural counterpart for Definition 4 as there is no notion of unit time step for continuous time systems. This leaves us with transition systems that either maintain timing information, or completely ignore it.

Definition 7 (Timed transition system $T_C^{\mathbb{R}_+}$). Consider continuous-time system C given by (10) and observation set $O = \mathbb{B}^p$. The transition system $T_C^{\mathbb{R}_+} = (Q, \Sigma, \to, O, \langle\langle\cdot\rangle\rangle)$ generated by C and O consists of:

- state space $Q = \mathbb{R}^n$,
- label $\Sigma = \mathbb{R}_+$,
- transition relation $\to \subseteq Q \times \mathbb{R}_+ \times Q$ defined as

$$x \xrightarrow{t} x' \Leftrightarrow \exists u_{[0,t]} \quad \text{with } x' = \Phi_C(t, x, u_{[0,t]}), \tag{12}$$

- observations $O = \mathbb{B}^p$,
- observation map $\langle\langle\cdot\rangle\rangle : Q \to O$ given by (5) and (6).

Similar to the k-step transitions of $T_\Delta^{\mathbb{N}_+}$, a transition $x \xrightarrow{t} x'$ of $T_C^{\mathbb{R}_+}$ captures the exact amount of time needed for the transition is maintained by the transition system. Conversely, the following transition system does not care about the exact timing information.

Definition 8 (Time-abstract transition system T_C^τ). Consider the continuous-time system C given by (10) and a set of observations $O = \mathbb{B}^p$. The time-abstract transition system $T_C^\tau = (Q, \Sigma, \to, O, \langle\langle\cdot\rangle\rangle)$ generated by C and O consists of:

- state space $Q = \mathbb{R}^n$,
- unique label $\Sigma = \{\tau\}$,
- transition relation $\to \subseteq Q \times \{\tau\} \times Q$ defined as

$$x \xrightarrow{\tau} x' \Leftrightarrow \exists t \in \mathbb{R}\ \exists u_{[0,t]} \quad \text{with } x' = \Phi_C(t, x, u_{[0,t]}), \tag{13}$$

- observations $O = \mathbb{B}^p$,
- observation map $\langle\langle\cdot\rangle\rangle : Q \to O$ given by (5) and (6).

In other words, a transition $x \xrightarrow{\tau} x'$ occurs if x' is reachable from x in any amount of time by a suitable choice of control.

5. Quotient transition systems

Having defined transition systems generated by discrete- and continuous-time systems allows us to proceed with the framework presented in Section 3 and consider in this section their quotients, and, in Section 6, their bisimulations.

5.1. Observation-preserving partitions

Given a partition $\sim$ of the state space, our first task is to construct the quotient transition system. In this paper, we shall partition $\mathbb{R}^n$ using surjective linear maps of the form $z = Hx$. Therefore $x_1 \sim x_2$ iff $Hx_1 = Hx_2$, and therefore the map $z = Hx$ is also the quotient map. Such a partition will be called a *linear partition* induced by the map $z = Hx$ or induced by matrix H. Since $H : \mathbb{R}^n \to \mathbb{R}^k$ is a full-row rank matrix, it is immediate that the states of the quotient system are $z \in \mathbb{R}^k$, resulting in partitions with an infinite number of states.[2]

In order to propagate the observations O and observation map $\langle\langle\cdot\rangle\rangle$ from the transition system to the quotient transition system, we must consider observation-preserving partitions. This will naturally place compatibility conditions between the observation map, and the linear partition map $z = Hx$. Consider our finite set of observations $O = \mathbb{B}^p$ with observation map $\langle\langle\cdot\rangle\rangle$ given by Eqs. (5) and (6). By definition, the partition is observation preserving if $x_1 \sim x_2$ implies that $\langle\langle x_1\rangle\rangle = \langle\langle x_2\rangle\rangle$, which is equivalent to $\langle\langle x_1\rangle\rangle_i = \langle\langle x_2\rangle\rangle_i$ for every predicate $1 \leqslant i \leqslant p$. The following proposition characterizes observation preserving, linear partitions for finite but affine observations.

Proposition 9 (Observation-preserving partitions). *Consider a linear partition of $\mathbb{R}^n$ induced by the surjective linear*

[2] For finite partitions of continuous-time linear systems, the reader is referred to Alur et al. (2000) and Lafferriere, Pappas, and Shankar Sastry (2000).

map $y = Hx$, *that is* $x_1 \sim x_2$ *iff* $Hx_1 = Hx_2$. *Consider an observation map* $\langle\langle\cdot\rangle\rangle : \mathbb{R}^n \to \mathbb{B}^p$ *given by equations* (5) and (6). *Then the linear partition is observation preserving if and only if*

$$\mathrm{Ker}(H) \subseteq \bigcap_{i=1}^{p} \mathrm{Ker}(a_i). \tag{14}$$

Proof. ($\Rightarrow$) Consider any $x_h \in \mathrm{Ker}(H)$. Since $x_h \sim 0$, and by assumption the partition is observation preserving, we have that $\langle\langle x_h\rangle\rangle = \langle\langle 0\rangle\rangle$ and therefore for all predicates $1 \leqslant i \leqslant p$, we have $\langle\langle x_h\rangle\rangle_i = \langle\langle 0\rangle\rangle_i$. Therefore for every predicate $a_i x_h + b_i \leqslant 0 \Leftrightarrow a_i \cdot 0 + b_i \leqslant 0 \Leftrightarrow b_i \leqslant 0$. Since b_i is a real constant, $b_i \leqslant 0$ is either true or false. Therefore, we must have that $a_i x_h + b_i \leqslant 0$ is either true or false for any $x_h \in \mathrm{Ker}(H)$. Since $\mathrm{Ker}(H)$ is a subspace, if $x_h \in \mathrm{Ker}(H)$ then $-\alpha x_h$ (for arbitrary scalar α) also belongs in $\mathrm{Ker}(H)$. Therefore we must have that $a_i x_h = 0$ and thus $x_h \in \mathrm{Ker}(a_i)$ for every predicate.

($\Leftarrow$) Consider any equivalent states $x_1 \sim x_2$, that is $Hx_1 = Hx_2$ and thus $x_2 = x_1 + x_h$ where $x_h \in \mathrm{Ker}(H) \subseteq \mathrm{Ker}(a_i)$ for all $1 \leqslant i \leqslant p$. But then for every $1 \leqslant i \leqslant p$ we have that $\langle\langle x_2\rangle\rangle_i = a_i x_2 + b_i = a_i(x_1 + x_h) + b_i = a_i x_1 + b_i + a_i x_h = a_i x_1 + b_i = \langle\langle x_1\rangle\rangle_i$. Since this is true for all $1 \leqslant i \leqslant p$, we have that $\langle\langle x_1\rangle\rangle = \langle\langle x_2\rangle\rangle$, hence the partition is observation preserving. □

As can be seen from condition (14), the more linearly independent row vectors a_i that are considered, the more information must be retained by the quotient system, and the more restrictive condition (14) becomes. Given an observation map $\langle\langle\cdot\rangle\rangle : \mathbb{R}^n \to \mathbb{B}^p$ and a linear observation preserving map $H : \mathbb{R}^n \to \mathbb{R}^k$, we can naturally define an observation map $\langle\langle\cdot\rangle\rangle_\sim : \mathbb{R}^k \to \mathbb{B}^p$. By definition, a state of the quotient system $z = Hx \in \mathbb{R}^k$ is mapped to observation o_i if there exists some $x \in H^{-1}(z)$ such that $\langle\langle x\rangle\rangle = o_i$. Since H is a full row rank matrix, we have that $H^{-1}(z) = H^+ z + \mathrm{Ker}(H)$ where H^+ is the Moore-Penrose pseudoinverse of H. Furthermore, since the partition is observation preserving, the observation is independent of the particular choice in $\mathrm{Ker}(H)$. We can therefore consider $0 \in \mathrm{Ker}(H)$ and define the observation map of the quotient system as

$$\langle\langle z\rangle\rangle_\sim = \langle\langle H^+ z\rangle\rangle. \tag{15}$$

From this point on, we will assume that our linear partitions are observation preserving.

5.2. *Construction of quotient transition system*

Given a transition system generated by a discrete- or continuous-time linear system as well as an observation preserving linear partition, the quotient transition needs to be defined. More importantly, we want to *construct* a linear system that *generates* the transitions of the quotient transition system.

5.2.1. *Discrete-time systems*

Let us begin by considering one-step transition systems $T^1_\Delta = (Q, \{1\}, \to, O, \langle\langle\cdot\rangle\rangle)$ generated by a discrete-time linear system Δ and a set of observations O. Consider an observation preserving partition $\sim$ induced by a linear surjective map $z = Hx$. The quotient transition system $T^1_\Delta/_\sim = (Q/_\sim, \{1\}, \to_\sim, O, \langle\langle\cdot\rangle\rangle_\sim)$ has $Q/_\sim = \mathbb{R}^k$, label set $\Sigma = \{1\}$, observation set $O = \mathbb{B}^p$, and observation map $\langle\langle\cdot\rangle\rangle_\sim$ defined as in the previous subsection. It remains to define the transition relation $\to_\sim$ to fully specify the quotient system $T^1_\Delta/_\sim$.

To define transitions of the quotient transition system, recall that there is a transition of $T^1_\Delta/_\sim$ from $z \xrightarrow{1}_\sim z'$ if there exists a transition of T^1_Δ from some $x \xrightarrow{1} x'$ with $z = Hx$ and $z' = Hx'$. Using the transition semantics from Definition 4, we know that transitions $x \xrightarrow{1} x'$ are generated by the discrete-time system $x' = Ax + Bu$ for some input u. It is reasonable to ask whether there exists a similar discrete-time linear system $z_{k+1} = Fz_k + Gv_k$ that *generates* the transitions of the quotient system $T^1_\Delta/_\sim$. More precisely, we ask that if there exists u such that $x \xrightarrow{1} x'$ then there exists a v such that $z \xrightarrow{1}_\sim z'$ with $z = Hx$ and $z' = Hx'$. Equivalently, for any $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, there must exist a $v \in \mathbb{R}^l$ such that $F(Hx) + Gv = H(Ax + Bu)$. The above discussion motivates the following definition which is the discrete-time version of the continuous-time definition found in Pappas et al. (2000).

Definition 10 (H-related linear systems). Consider the discrete-time linear control systems

$$\Delta_1: \quad x_{k+1} = Ax_k + Bu_k, \quad x \in \mathbb{R}^n, \; u \in \mathbb{R}^m,$$

$$\Delta_2: \quad z_{k+1} = Fz_k + Gv_k, \quad z \in \mathbb{R}^k, \; v \in \mathbb{R}^l$$

and the linear, surjective map $z = Hx$. Then Δ_2 is H-related to Δ_1 if for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, there exists $v \in \mathbb{R}^l$ such that

$$H(Ax + Bu) = FHx + Gv.$$

Therefore Δ_2 is H-related to Δ_1 if at every state x and for any control move u of Δ_1, there exists a matching move v of Δ_2 such that $z = Hx$. The following proposition gives us a method for *constructing* H-related linear systems.[3]

Proposition 11 (Canonical construction (Pappas et al., 2000)). *Consider the linear system*

$$\Delta_1: \quad x_{k+1} = Ax_k + Bu_k$$

and a surjective map $z = Hx$. *Let*

$$\Delta_2: \quad z_{k+1} = Fz_k + Gv_k$$

[3] For nonlinear generalizations of this construction, the reader is referred to Pappas and Simic (2002). A notion that is related to the notion of H-related linear systems can be found in Stankovic and Siljak (2002).

be the system where

$F = HAH^{+},$

$G = [HB\ HAw_1 \cdots HAw_r],$

where H^{+} *is the Moore–Penrose pseudoinverse of* H, *and* $w_1,\ldots,w_r$ span $\mathrm{Ker}(H)$. *Then* Δ_2 *is* H*-related to* Δ_1.

Note that by Proposition 11, given any discrete time linear control system and any full-row rank matrix H, there *always* exists another discrete-time linear control system which is H-related to it. Furthermore, given Δ_1 there is a *constructive* method for computing the H-related system Δ_2. Therefore, if Δ_1 is the discrete-time system that generates the transitions of $T^1_{\Delta_1}$, then Δ_2 can generate the transitions of the quotient system $T^1_{\Delta_1}/\sim$ where $\sim$ is the partition induced by the map $z=Hx$. With this construction, the quotient transition system $T^1_{\Delta}/\sim$ is fully specified.

For transition systems $T^{\mathbb{N}_+}_{\Delta}/\sim$ and $T^{\tau}_{\Delta}/\sim$ everything is the same as in $T^1_{\Delta}/\sim$, the only exception being the definition of the transition relation as well as the linear systems that generate those transitions. However, since one-step transitions are included in k-step transitions, the linear system that generates the transitions of $T^{\mathbb{N}_+}_{\Delta}/\sim$ must be at least H-related to the linear system that generates the transitions of $T^{\mathbb{N}_+}_{\Delta}$. The following proposition shows that this is also sufficient.

Theorem 12 (Trajectory propagation). *Consider the discrete-time linear systems*

$$\Delta_1: \quad x_{k+1} = Ax_k + Bu_k,$$

$$\Delta_2: \quad z_{k+1} = Fz_k + Gv_k$$

and the linear, surjective map $z=Hx$. *Then* Δ_2 *produces as state trajectories all sequences of the form* $\{z_i\}_{i=0}^{k}$, *where for all* $0 \leqslant i \leqslant k$ *we have* $z_i=Hx_i$ *and* $\{x_i\}_{i=0}^{k}$ *is a trajectory of* Δ_1, *if and only if* Δ_2 *is* H*-related to* Δ_1.

Proof. Consider input sequence $\{u_i\}_{i=0}^{k-1}$ resulting in state sequence $\{x_i\}_{i=0}^{k}$, where $x_{i+1} = Ax_i + Bu_i$ for $0 \leqslant i \leqslant k-1$. Since Δ_2 is H-related to Δ_1, for any $0 \leqslant i \leqslant k-1$ and each evolution $x_{i+1} = Ax_i + Bu_i$ we have that there exists v_i such that $z_{i+1} = Hx_{i+1} = F(Hx_i) + Gv_i$. Therefore, there exists $\{v_i\}_{i=0}^{k-1}$ resulting in state trajectory $\{z_i\}_{i=0}^{k}$ where $z_i = Hx_i$. Necessity is immediate by simply considering trajectories with $k=1$. □

By the above theorem, if $x\xrightarrow{k}x'$ using discrete-time system Δ_1, then $Hx\underset{\sim}{\xrightarrow{k}}Hx'$ using discrete-time system Δ_2 . Therefore, Δ_2 can be used to generate the transitions of the quotient transition system $T^{\mathbb{N}_+}_{\Delta_1}/\sim$. Furthermore, the following corollary shows that Δ_2 can also be used to generate the transitions of time-abstract transition system $T^{\tau}_{\Delta_1}/\sim$.

Corollary 13 (Reachability propagation). *Consider the discrete-time linear systems*

$$\Delta_1: \quad x_{k+1} = Ax_k + Bu_k,$$

$$\Delta_2: \quad z_{k+1} = Fz_k + Gv_k,$$

where Δ_2 *is* H*-related to* Δ_1 *with respect to the surjective map* $z=Hx$. *Then, if* x' *is reachable from* x *with a* Δ_1 *trajectory, then* $z' = Hx'$ *is reachable from* $z = Hx$ *with a* Δ_2 *trajectory. Furthermore, if* Δ_1 *is controllable, then* Δ_2 *is controllable.*

The above results allow us to use discrete-time system Δ_2 as the generator for the transitions of the quotient systems $T^1_{\Delta_1}/\sim$, $T^{\mathbb{N}_+}_{\Delta_1}/\sim$, and $T^{\tau}_{\Delta_1}/\sim$, for all transition systems that were generated by the original discrete-time system Δ_1.

5.2.2. Continuous-time systems

For continuous-time systems, the definition of the quotient transition systems $T^{\mathbb{R}_+}_{C}/\sim$ and $T^{\tau}_{C}/\sim$ follows the same path as for the transition systems generated by discrete-time systems. The only thing that changes from the discrete world is the continuous-time generators of the transitions of the quotient transition system. The continuous-time version of H-related systems was originally obtained in Pappas et al. (2000) in the context of controllability preserving abstractions of linear systems.

Definition 14 (H-related linear systems). Consider the continuous-time linear control systems

$$C_1: \quad \dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}^m,$$

$$C_2: \quad \dot{z} = Fz + Gv, \quad z \in \mathbb{R}^k, \ v \in \mathbb{R}^l$$

and the linear, surjective map $z=Hx$. Then C_2 is H-related to C_1 iff for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, there exists $v \in \mathbb{R}^l$ such that

$$H(Ax+Bu) = FHx + Gv.$$

Therefore, system C_2 must be able to generate using control input $v \in \mathbb{R}^l$, any tangent vector that system C_1 may generate at any point $x \in \mathbb{R}^n$, and given any control input $u \in \mathbb{R}^m$. Notice that since the matrix conditions in Definitions 14 and 10 are the same, the construction of Proposition 11 applies to continuous-time as well.

Theorem 15 (Trajectory propagation (Pappas et al., 2000)). *Consider the linear time-invariant control systems*

$$C_1: \quad \dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}^m,$$

$$C_2: \quad \dot{z} = Fz + Gv, \quad z \in \mathbb{R}^k, \ v \in \mathbb{R}^l$$

and the linear, surjective map $z = Hx$. *Then* C_2 *produces as state trajectories all functions of the form* $z(t) = Hx(t)$, *where* $x(t)$ *is a trajectory of* C_1, *if and only if* C_2 *is* H*-related to* C_1.

By the above theorem, if $x\xrightarrow{t}x'$, then $Hx\xrightarrow{t}{\sim}Hx'$, and therefore, C_2 can be used to generate the transitions of the quotient transition system $T^{\mathbb{R}_+}_{C_1}/\sim$ as well as $T^{\tau}_{C_1}/\sim$.

6. Bisimilar linear systems

Up to this point, we have defined a variety of transition systems generated by linear systems, and have constructed linear system generators for quotient transition systems given observation preserving linear partitions. It remains to characterize equivalences that have the bisimulation property. Since the bisimulation property depends on the set of labels Σ, the characterizations will be different for different transition systems.

In all cases, however, we shall rely on Proposition 3 which considered transition system T, observation-preserving equivalence relation $\sim$, quotient map $h : Q \to Q/_\sim$, and required that for all states $q \in Q$ and for all $\sigma \in \Sigma$, that

$$h(Post_\sigma(h^{-1}(h(q)))) = h(Post_\sigma(q)). \tag{16}$$

For both discrete- and continuous-time systems, the quotient map is $h(x)=Hx$, thus $h^{-1}(h(x))=x+\mathrm{Ker}(H)$. Therefore, in order for a linear partition induced by H to be a bisimulation, the following condition must be satisfied for all $x \in \mathbb{R}^n$ and for all $\sigma \in \Sigma$.

$$H(Post_\sigma(x + \mathrm{Ker}(H))) = H(Post_\sigma(x)). \tag{17}$$

What will be different for each transition system we have considered will be the label set Σ, and the operators $Post_\sigma(\cdot)$. We begin with the transition systems considered for discrete-time systems.

6.1. Discrete-time systems

We begin by first rephrasing the definition of bisimulation in the context of T^1_Δ, where there is only one label, that is $\Sigma = \{1\}$. Consider any $x_1 \sim x_2$ or equivalently $Hx_1 = Hx_2$ for full row rank matrix H. The bisimulation property asks the following: If there exists a control u_1 such that $x_1 \xrightarrow{1} x_1' \Rightarrow x_1' = Ax_1 + Bu_1$ then there must exist a control u_2 such that $x_2 \xrightarrow{1} x_2' \Rightarrow x_2' = Ax_2 + Bu_2$ with $x_1' \sim x_2'$ or equivalently $Hx_1' = Hx_2'$. Equivalently, we can use condition (17), where the operator

$$Post_1(x) = Ax + \mathcal{R}(B) = Ax + \mathrm{span}\{B\} \tag{18}$$

captures the set that is reachable from x in one time step.

Theorem 16 (Bisimulations of T^1_Δ). *Consider transition system T^1_Δ generated by the discrete-time linear system Δ and observation set $O = \mathbb{B}^p$. Consider an observation preserving partition induced by the surjective, linear map $z = Hx$, that is $x_1 \sim x_2$ iff $Hx_1 = Hx_2$. The partition is a bisimulation if and only if*

$$A\,\mathrm{Ker}(H) \subseteq \mathrm{Ker}(H) + \mathcal{R}(B). \tag{19}$$

Proof. Condition (17) along with (18) require that for all $x \in \mathbb{R}^n$ we satisfy

$$H(A(x + \mathrm{Ker}(H)) + \mathcal{R}(B)) = H(Ax + \mathcal{R}(B)). \tag{20}$$

($\Leftarrow$) Assuming (19) holds, we show that condition (20) holds. The inclusion

$$H(A(x + \mathrm{Ker}(H)) + \mathcal{R}(B)) \supseteq H(Ax + \mathcal{R}(B))$$

is clear. Condition (19) results in

$$\begin{aligned} &H(A(x + \mathrm{Ker}(H)) + \mathcal{R}(B)) \\ &\quad = HAx + HA\,\mathrm{Ker}(H) + H\mathcal{R}(B) \\ &\quad \subseteq HAx + H(\mathrm{Ker}(H) + \mathcal{R}(B)) \subseteq H(Ax + \mathcal{R}(B)), \end{aligned}$$

which proves the other inclusion of condition (20).

($\Rightarrow$) Conversely assume that condition (20) holds for all $x \in \mathbb{R}^n$. Since condition (20) must hold for all $x \in \mathbb{R}^n$, then it must also be true for $x = 0$, in which case, we get that

$$H(A\,\mathrm{Ker}(H) + \mathcal{R}(B)) = H(\mathcal{R}(B)). \tag{21}$$

Consider any $x_h \in \mathrm{Ker}(H)$ where $x_h \sim 0$ since $Hx_h = H \cdot 0 = 0$. From the above condition we obtain that for all $x_h \in \mathrm{Ker}(H)$, there exists $b_h \in \mathcal{R}(B)$ such that $HAx_h = Hb_h$. But then $Ax_h = x_h' + b_h$ where $x_h' \in \mathrm{Ker}(H)$ and $b_h \in \mathcal{R}(B)$, thus proving condition (19). □

Therefore, the main condition (19) for the partition to have the bisimulation property requires $\mathrm{Ker}(H)$ to be a *controlled invariant* subspace (Wonham, 1985). This result clearly establishes a closer connection between well known concepts from control theory (controlled invariance in Wonham, 1985) and theoretical computer science (bisimulation in Milner, 1989).

When considering transition system $T^{\mathbb{N}_+}_\Delta$, condition (17) must be used with the following operators:

$$Post_k(x) = A^k x + Post_k(0), \tag{22}$$

$$Post_k(0) = \bigoplus_{i=0}^{k-1} \mathcal{R}(A^i B) = \mathrm{span}\{B\ AB \cdots A^{k-1}B\} \tag{23}$$

where $Post_k(x)$ contains all states that can be reached from $x \in \mathbb{R}^n$ in k steps, whereas $Post_k(0)$ contains all states that can be reached from the origin in k steps. Because of the Cayley–Hamilton theorem, for $k \geqslant n$ matrix A^k is a linear combination of $I, A, A^2, \ldots, A^{n-1}$, thus resulting in the following operators if $k \geqslant n$,

$$Post_k(x) = A^k x + \mathcal{R}(A, B), \tag{24}$$

$$Post_k(0) = \mathcal{R}(A, B), \tag{25}$$

where $\mathcal{R}(A,B) = \mathrm{span}\{B\ AB \cdots A^{n-1}B\}$ is the controllability subspace. The following corollary of Theorem 16 shows that controlled invariance of $\mathrm{Ker}(H)$ is also what is needed for transition systems $T^{\mathbb{N}_+}_\Delta$ and $T^{\mathbb{N}_+}_\Delta/_\sim$ to be bisimilar.

Corollary 17 (Bisimulations of $T^{\mathbb{N}_+}_\Delta$). *Consider transition system $T^{\mathbb{N}_+}_\Delta$ generated by the discrete-time linear system Δ and observation set $O = \mathbb{B}^p$. Consider an observation preserving partition induced by the surjective,* *linear map $z = Hx$. Then the partition is a bisimulation if and only if*

$$A\,\mathrm{Ker}(H) \subseteq \mathrm{Ker}(H) + \mathscr{R}(B). \tag{26}$$

Proof. In the context of transition system $T_\Delta^{\mathbb{N}_+}$ where $\Sigma = \mathbb{N}_+$, condition (17) and operator (22) require that for all $k \geqslant 0$ and for all $x \in \mathbb{R}^n$ we satisfy

$$H\left(A^k(x + \mathrm{Ker}(H)) + \bigoplus_{i=0}^{k-1} \mathscr{R}(A^i B)\right) = H\left(A^k x + \bigoplus_{i=0}^{k-1} \mathscr{R}(A^i B)\right). \tag{27}$$

($\Rightarrow$) Since (27) must be true for all $k \geqslant 0$, it must also be true for $k = 1$, in which case (27) reduces to (20) which implies (26) from Theorem 16.

($\Leftarrow$) Assuming condition (26), a simple induction argument shows that

$$A^k\,\mathrm{Ker}(H) \subseteq \mathrm{Ker}(H) + \bigoplus_{i=0}^{k-1} \mathscr{R}(A^i B), \tag{28}$$

which proves the nontrivial inclusion of (27). □

As the time-abstract transition system T_Δ^τ ignores all timing information, the condition for bisimulation should be naturally weaker. The relevant operator in this case is

$$Post_\tau(x) = \bigcup_{k \geqslant 0} A^k x + \mathscr{R}(A, B), \tag{29}$$

which captures the reachable set from any point $x \in \mathbb{R}^n$.

Theorem 18 (Bisimulations of T_Δ^τ). *Consider transition system T_Δ^τ generated by the discrete-time linear system Δ and observation set $O = \mathbb{B}^p$. Consider an observation preserving partition induced by the surjective, linear map $z = Hx$, that is $x_1 \sim x_2$ iff $Hx_1 = Hx_2$. The partition is a bisimulation if and only if*

$$A\,\mathrm{Ker}(H) \subseteq \mathrm{Ker}(H) + \mathscr{R}(A, B). \tag{30}$$

Proof. Condition (17) and operator (29) require that for all $x \in \mathbb{R}^n$ we have

$$H\left(\bigcup_{k \geqslant 0} A^k(x + \mathrm{Ker}(H)) + \mathscr{R}(A, B)\right) = H\left(\bigcup_{k \geqslant 0} A^k x + \mathscr{R}(A, B)\right). \tag{31}$$

($\Leftarrow$) Assuming (30) holds, we show that condition (31) holds. One inclusion is obvious. Condition (30) and A-invariance of $\mathscr{R}(A, B)$ imply that for all $k \geqslant 0$ we have $A^k\mathrm{Ker}(H) \subseteq \mathrm{Ker}(H) + \mathscr{R}(A, B)$ and therefore for all $k \geqslant 0$, $HA^k\mathrm{Ker}(H) \subseteq H\mathscr{R}(A, B)$ which proves the other inclusion of condition (31).

($\Rightarrow$) Conversely assume that condition (31) holds for all $x \in \mathbb{R}^n$. Consider any $x_h \in \mathrm{Ker}(H)$ where $x_h \sim 0$ since $Hx_h = H \cdot 0 = 0$. Since condition (31) must hold for all $x \in \mathbb{R}^n$, then it must also be true for $x = 0$, in which case therefore, we get that for any $k \geqslant 0$ there exists $r \in \mathscr{R}(A, B)$ such that $HA^k x_h = Hr$. Therefore, $A^k x_h = x'_h + r$ for some $x'_h \in \mathrm{Ker}(H)$. Since this is true for any $k \geqslant 0$, then it is also true for $k = 1$. Therefore, $Ax_h \in \mathrm{Ker}(H) + \mathscr{R}(A, B)$. □

The above result motivates us to define a subspace $\mathscr{V}$ as *reachability invariant* if

$$A\mathscr{V} \subseteq \mathscr{V} + \mathscr{R}(A, B) = \mathscr{V} + \mathrm{span}\{B\,AB \cdots A^{n-1}B\}. \tag{32}$$

Condition (30) for bisimilar time-abstract transition systems therefore requires $\mathrm{Ker}(H)$ to be reachability invariant. Note that reachability invariance is clearly much weaker than the control invariance condition (19). In fact, reachability invariance can always be satisfied by choosing matrix H with $\mathrm{Ker}(H) \subseteq \mathscr{R}(A, B)$ or even $\mathrm{Ker}(H) \subseteq \mathscr{R}(B)$. Furthermore, if the system is controllable, then condition (30) is automatically satisfied for any H, in which case observational equivalence immediately results in bisimulation.

6.2. Continuous-time systems

We begin with timed transition system $T_C^{\mathbb{R}_+}$ where the operator

$$\mathrm{Post}_t(x) = \mathrm{e}^{At}x + \mathscr{R}(A, B) \tag{33}$$

describes the reachable set from x at exactly time $t \in \mathbb{R}$.

Theorem 19 (Bisimulations of $T_C^{\mathbb{R}_+}$). *Consider transition system $T_C^{\mathbb{R}_+}$ generated by the continuous-time linear system C and observation set $O = \mathbb{B}^p$. Consider an observation preserving partition induced by the surjective, linear map $z = Hx$, that is $x_1 \sim x_2$ iff $Hx_1 = Hx_2$. The partition is a bisimulation if and only if*

$$A\,\mathrm{Ker}(H) \subseteq \mathrm{Ker}(H) + \mathscr{R}(A, B). \tag{34}$$

Proof. Condition (17) combined with operator (33) require that for any $t \in \mathbb{R}_+$ and for any $x \in \mathbb{R}^n$ we have

$$H(\mathrm{e}^{At}(x + \mathrm{Ker}(H)) + \mathscr{R}(A, B)) = H(\mathrm{e}^{At}x + \mathscr{R}(A, B)). \tag{35}$$

($\Leftarrow$) Condition (34) and the fact that $\mathscr{R}(A, B)$ is A-invariant, imply that for any $t \in \mathbb{R}_+$ we have

$$\mathrm{e}^{At}\,\mathrm{Ker}(H) \subseteq \mathrm{Ker}(H) + \mathscr{R}(A, B),$$

$$H\mathrm{e}^{At}\,\mathrm{Ker}(H) \subseteq H\mathscr{R}(A, B), \tag{36}$$

which directly implies the nontrivial inclusion of (35).

($\Rightarrow$) Assume that condition (35) holds for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$. Since condition (35) must hold for all $x \in \mathbb{R}^n$, then it must also be true for $x = 0$, in which case therefore, we get that

$$H(\mathrm{e}^{At}\,\mathrm{Ker}(C) + \mathscr{R}(A, B)) = H(\mathscr{R}(A, B)).$$

Table 1
Algebraic characterizations of bisimulation

Transition system	Bisimulation condition
T_Δ^1	$A\,\mathrm{Ker}(H) \subseteq \mathrm{Ker}(H) + \mathscr{R}(B)$
$T_\Delta^{\mathbb{N}_+}$	$A\,\mathrm{Ker}(H) \subseteq \mathrm{Ker}(H) + \mathscr{R}(B)$
T_Δ^τ	$A\,\mathrm{Ker}(H) \subseteq \mathrm{Ker}(H) + \mathscr{R}(A,B)$
$T_C^{\mathbb{R}_+}$	$A\,\mathrm{Ker}(H) \subseteq \mathrm{Ker}(H) + \mathscr{R}(A,B)$
T_C^τ	$A\,\mathrm{Ker}(H) \subseteq \mathrm{Ker}(H) + \mathscr{R}(A,B)$

Consider any $x_h \in \mathrm{Ker}(H)$ where $x_h \sim 0$ since $Hx_h = H \cdot 0 = 0$. Therefore, for all t there exists $r \in \mathscr{R}(A,B)$ such that $He^{At}x_h = Hr$. Therefore, $e^{At}x_h = x_h' + r$ for some $x_h' \in \mathrm{Ker}(H)$. Since for all t we have that $e^{At}x_h \in \mathrm{Ker}(H) + \mathscr{R}(A,B)$ then by differentiating $de^{At}/dt = Ae^{At}$ and taking the limit as $t \to 0$, we conclude that $Ax_h \in \mathrm{Ker}(H) + \mathscr{R}(A,B)$. □

Since $T_C^{\mathbb{R}_+}$ is the continuous-time counterpart of $T_\Delta^{\mathbb{N}_+}$, one would have expected conditions (34) and (26) to be the same. However, condition (34) for continuous time is clearly weaker than condition (26) for discrete time. The main reason for this difference is the existence of an atomic unit of time for discrete-time systems. The lack of such a notion for continuous-time systems allows us to reach states in *any* time, if we can reach them at all.

This close relationship between timed transition system $T_C^{\mathbb{R}_+}$ and time-abstract transition system T_C^τ is also highlighted by the following theorem and which was originally proven in Pappas et al. (2000) in the context of controllability preserving abstractions of continuous-time linear systems.

Theorem 20 (Bisimulations of T_C^τ). *Consider transition system T_C^τ generated by the continuous-time linear system C and observation set $O = \mathbb{B}^p$. Consider an observation preserving partition induced by the surjective, linear map $z = Hx$, that is $x_1 \sim x_2$ iff $Hx_1 = Hx_2$. The partition is a bisimulation if and only if*

$$A\,\mathrm{Ker}(H) \subseteq \mathrm{Ker}(H) + \mathscr{R}(A,B). \tag{37}$$

Table 1 summarizes the bisimulation characterization for all transition systems that were considered. Each of the following conditions, must be complemented by the conditions that result in observational equivalence for finite but affine observations (14).

It is natural to search for the quotient map that abstracts away as much as possible while resulting in bisimilar transition systems. This is the focus of the next section.

7. Fixed point characterization

Bisimulation is often viewed as the *coarsest* partition that refines observational equivalence while satisfying the condition that equivalent states can reach equivalent states using the same label. This view of bisimulation leads to a fixed point characterization of bisimulation as well as a bisimulation algorithm. The algorithm starts with observational equivalence, and appropriately refines the partition until it reaches a fixed point which is the coarsest bisimulation (Alur et al., 2000; Milner, 1989).

In the context of transition systems generated by linear systems, we are searching for a quotient map $z = Hx$, with maximal $\mathrm{Ker}(H)$, that refines observational equivalence (for either linear or finite observations), and satisfies one of the conditions of Table 1. For all transition systems considered, refining observational equivalence required that $\mathrm{Ker}(H) \subseteq \bigcap_{i=1}^p \mathrm{Ker}(a_i)$. From Table 1, the bisimulation property for transition systems T_Δ^1 and $T_\Delta^{\mathbb{N}_+}$ required that

$$A\mathrm{Ker}(H) \subseteq \mathrm{Ker}(H) + \mathscr{R}(B). \tag{38}$$

Therefore, finding the coarsest partition, requires finding the *maximal* controlled invariant subspace that lives inside $\bigcap_{i=1}^p \mathrm{Ker}(a_i)$. The following well-known result from the geometric theory of linear systems provides us with an algorithmic solution to this problem.

Theorem 21 (Controlled controlled subspaces in Wonham, 1985). *Let $\mathscr{K}$ be any subspace in $\mathbb{R}^n$. Then there exists a unique, maximal controlled invariant subspace $\mathscr{V}^*$ contained in $\mathscr{K}$ and satisfying*

$$A\mathscr{V}^* \subseteq \mathscr{V}^* + \mathscr{R}(B). \tag{39}$$

Define the sequence of subspaces of $\mathscr{K}$

$$\mathscr{V}^0 = \mathscr{K} \tag{40}$$

$$\mathscr{V}^k = \mathscr{V}^{k-1} \cap A^{-1}(\mathscr{V}^{k-1} + \mathscr{R}(B)), \quad k = 1, 2, \ldots. \tag{41}$$

Then $\mathscr{V}^n = \mathscr{V}^*$ is the unique, maximal, controlled invariant subspace contained in $\mathscr{K}$.

The iteration of Theorem 21 with $\mathscr{V}^0 = \bigcap_{i=1}^p \mathrm{Ker}(a_i)$ can be used as our bisimulation algorithm in order to compute, in at most n steps, the desired map $z = Hx$. The coarsest bisimulation for transition systems T_Δ^1 and $T_\Delta^{\mathbb{N}_+}$, can then be obtained by simply choosing quotient map $z = Hx$ with $\mathrm{Ker}(H) = \mathscr{V}^*$.

In the case of transition systems T_Δ^τ, $T_C^{\mathbb{R}_+}$, and T_C^τ, in addition to refining observational equivalence, partitions must be reachability invariant, and thus satisfy

$$A\,\mathrm{Ker}(H) \subseteq \mathrm{Ker}(H) + \mathscr{R}(A,B). \tag{42}$$

We are therefore interested in finding the *maximal* reachability invariant subspace inside $\bigcap_{i=1}^p \mathrm{Ker}(a_i)$ that satisfies condition (42). In contrast to the well studied controlled invariant subspaces, there is no known fixed point characterization of the maximal reachability invariant subspace living inside $\bigcap_{i=1}^p \mathrm{Ker}(a_i)$ that satisfies (42). The next result establishes such a characterization. The proof follows the lines of the proof of Theorem 21, and may be found in (Pappas, 2001).

Theorem 22 (Reachability invariant subspace). *Let $\mathscr{K}$ be any subspace in $\mathbb{R}^n$. Then there exists a unique, maximal subspace $\mathscr{V}^*$ contained in $\mathscr{K}$ and satisfying*

$$A\mathscr{V}^* \subseteq \mathscr{V}^* + \mathscr{R}(A,B). \tag{43}$$

Define the sequence of subspaces of $\mathscr{K}$ as

$$\mathscr{V}^0 = \mathscr{K}$$

$$\mathscr{V}^k = \mathscr{K} \cap A^{-1}(\mathscr{V}^{k-1} + \mathscr{R}(A,B)), \quad k = 1,2,\ldots . \tag{44}$$

Then $\mathscr{V}^n = \mathscr{V}^*$ is the unique, maximal, subspace contained in $\mathscr{K}$ and satisfying

$$A\mathscr{V}^n \subseteq \mathscr{V}^n + \mathscr{R}(A,B). \tag{45}$$

The iteration of Theorem 22 with $\mathscr{V}^0 = \bigcap_{i=1}^p \mathrm{Ker}(a_i)$ can be used as our bisimulation algorithm in order to compute the quotient map $z = Hx$. Then the coarsest bisimulation for transition systems T_Δ^τ, $T_C^{\mathbb{R}_+}$, and T_C^τ can be obtained by choosing quotient map $z = Hx$ with $\mathrm{Ker}(H) = \mathscr{V}^*$.

8. Special cases

8.1. Bisimulations and exact model reduction

Let us begin with a special example that is particularly interesting as it shows that complexity reduction ideas from theoretical computer science and control theory are closer than expected. Consider transition systems T_Δ^1 and $T_\Delta^{\mathbb{N}_+}$ generated by discrete-time system

$$\Delta: \quad x_{k+1} = Ax_k + Bu_k, \tag{46}$$

but with linear observations $\langle\langle x \rangle\rangle = Cx$. It is straightforward to show that linear quotient maps $z = Hx$ preserve the observations if and only if $\mathrm{Ker}(H) \subseteq \mathrm{Ker}(C)$. Furthermore, the map $z = Hx = Cx$ induces a bisimulation if and only if

$$A\,\mathrm{Ker}(C) \subseteq \mathrm{Ker}(C) + \mathscr{R}(B). \tag{47}$$

Assuming $\mathrm{Ker}(C)$ is controlled invariant, the generator of the bisimilar quotient transition systems $T_\Delta^1/_\sim$ and $T_\Delta^{\mathbb{N}_+}/_\sim$ is the C-related discrete-time system $y_{k+1} = Fy_k + Gv_k$, where the construction of Proposition 11 results in $F = CAC^+, G = [CB\ CAK\mathrm{er}(C)]$. If we compose the bisimulation condition (47) with the above construction, we obtain that $\mathscr{R}(G) = \mathscr{R}(CB) + CA\,\mathrm{Ker}(C) \subseteq \mathscr{R}(CB) + C(\mathrm{Ker}(C) + \mathscr{R}(B)) \subseteq \mathscr{R}(CB) + C\mathscr{R}(B) \subseteq \mathscr{R}(CB)$. Therefore, we can simply choose $G = CB$ in our canonical construction. The generator of $T_\Delta^1/_\sim$ and $T_\Delta^{\mathbb{N}_+}/_\sim$ is thus the C-related discrete-time system

$$y_{k+1} = CAC^+ y_k + CBu_k. \tag{48}$$

Note that the above model has the same input as the original system Δ. In fact, the above model is exactly the model that is obtained using *exact model reduction* techniques in control theory that go as far back as (Aoki, 1968). This result should be expected as models (and therefore model reduction techniques) in computer science are *exact*. Even though exact model reduction is a strong requirement for control systems, the above result clearly connects complexity reduction methods from computer science and control theory. The more popular approximate model reduction techniques would not result in *exact* observational equivalence for linear observations. However, exact observational equivalence may still be possible for finite observations. Further research must explore approximations of control systems, that nonetheless lead to exact observational equivalence with respect to a finite set of observations.

8.2. Reachability equivalence with finite observations

We now show by example some of the computations by considering linear systems with finite polyhedral observations. Consider the linear system $\dot{x} = Ax + Bu$ where

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \dot{x}_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u \tag{49}$$

and the observation map $\langle\langle\cdot\rangle\rangle : \mathbb{R}^4 \to \mathbb{B}^6$ is generated by the following six predicates

$$\langle\langle x \rangle\rangle_1 = \begin{cases} 1 & \text{if } x_1 + 2 \leqslant 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$\langle\langle x \rangle\rangle_2 = \begin{cases} 1 & \text{if } -x_1 + 2 \leqslant 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$\langle\langle x \rangle\rangle_3 = \begin{cases} 1 & \text{if } x_4 + 2 \leqslant 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$\langle\langle x \rangle\rangle_4 = \begin{cases} 1 & \text{if } -x_4 + 2 \leqslant 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$\langle\langle x \rangle\rangle_5 = \begin{cases} 1 & \text{if } -x_1 - x_4 - 10 \leqslant 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$\langle\langle x \rangle\rangle_6 = \begin{cases} 1 & \text{if } x_1 + x_4 - 10 \leqslant 0, \\ 0 & \text{otherwise.} \end{cases}$$

A natural reachability question is to ask whether we can observe output $(0,1,0,1,0,1)$ at any time after observing $(1,0,1,0,1,0)$. This is equivalent to the reachability problem of whether we can reach, at any time, the set $(x_1 \geqslant 2) \wedge (x_4 \geqslant 2) \wedge (x_1 + x_4 \leqslant 10)$ from the initial set $(x_1 \leqslant -2) \wedge (x_4 \leqslant -2) \wedge (x_1 + x_4 \geqslant -10)$. Since we are interested in time-abstract reachability, we shall consider T_C^τ, the time-abstract transition system generated by system (49) with the above discrete observations.

Our goal is to compress T_C^τ as much as possible using linear quotient maps while generating the same sequence of predicates. We must therefore consider quotient maps $z=Hx$ that refine observational equivalence, and thus satisfy $\mathrm{Ker}(H) \subseteq \bigcap_{i=1}^{6} \mathrm{Ker}(a_i)$, where the relevant row vectors a_i for $i = 1, \ldots, 6$ can be seen to be

$$a_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \qquad a_2 = \begin{bmatrix} -1 & 0 & 0 & 0 \end{bmatrix},$$

$$a_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}, \qquad a_4 = \begin{bmatrix} 0 & 0 & 0 & -1 \end{bmatrix},$$

$$a_5 = \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}, \qquad a_6 = \begin{bmatrix} -1 & 0 & 0 & -1 \end{bmatrix}.$$

Preserving the observations requires that

$$\mathrm{Ker}(H) \subseteq \bigcap_{i=1}^{6} \mathrm{Ker}(a_i) = \mathrm{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}, \tag{50}$$

which simply means that the x_2 and x_3 directions can be ignored without affecting the observations. We therefore choose the quotient map

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = Hx = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}. \tag{51}$$

We must then check whether the bisimulation condition for T_C^τ is satisfied. The controllability subspace is

$$\mathscr{R}(A,B) = \mathrm{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix} \right\}, \tag{52}$$

which is clearly three dimensional and thus the system is not controllable. However, since $\mathrm{Ker}(H) \subseteq \mathscr{R}(A,B)$ we have that $\mathrm{Ker}(H)$ is reachability invariant and thus the bisimulation condition for T_C^τ is indeed satisfied. We can therefore proceed with Proposition 11 and construct the H-related linear system $\dot{z} = Fz + Gv$ where the computations result in

$$\dot{z} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v = Fz + Gv. \tag{53}$$

The above linear system generates the transitions of quotient system $T_C^\tau/_\sim$ which is bisimilar to T_C^τ. The observation map $\langle\langle \cdot \rangle\rangle_\sim : \mathbb{R}^2 \to \mathbb{B}^6$ is defined as $\langle\langle z \rangle\rangle_\sim = \langle\langle H^+ z \rangle\rangle$ resulting in the following predicates

$$\langle\langle z \rangle\rangle_{\sim_1} = \begin{cases} 1 & \text{if } z_1 + 2 \leqslant 0, \\ 0 & \text{otherwise}, \end{cases}$$

$$\langle\langle z \rangle\rangle_{\sim_2} = \begin{cases} 1 & \text{if } -z_1 + 2 \leqslant 0, \\ 0 & \text{otherwise}, \end{cases}$$

$$\langle\langle z \rangle\rangle_{\sim_3} = \begin{cases} 1 & \text{if } z_2 + 2 \leqslant 0, \\ 0 & \text{otherwise}, \end{cases}$$

$$\langle\langle z \rangle\rangle_{\sim_4} = \begin{cases} 1 & \text{if } -z_2 + 2 \leqslant 0, \\ 0 & \text{otherwise}, \end{cases}$$

$$\langle\langle z \rangle\rangle_{\sim_5} = \begin{cases} 1 & \text{if } -z_1 - z_2 - 10 \leqslant 0, \\ 0 & \text{otherwise}, \end{cases}$$

$$\langle\langle z \rangle\rangle_{\sim_6} = \begin{cases} 1 & \text{if } z_1 + z_2 - 10 \leqslant 0, \\ 0 & \text{otherwise}. \end{cases}$$

We can, therefore, equivalently solve the following reachability problem on the reduced second-order system: can we reach, at any time, the set $(z_1 \geqslant 2) \wedge (z_2 \geqslant 2) \wedge (z_1 + z_2 \leqslant 10)$ from the initial set $(z_1 \leqslant -2) \wedge (z_2 \leqslant -2) \wedge (z_1 + z_2 \geqslant -10)$, which can be seen to be true.

In the above example, we were fortunate since $\mathrm{Ker}(H) = \bigcap_{i=1}^{6} \mathrm{Ker}(a_i) \subseteq \mathscr{R}(A,B)$, and therefore observational equivalence was sufficient. If, however, the input matrix B in (49) changes to $B = [1\ 0\ 1\ 0]^{\mathrm{T}}$, then we obtain that

$$\mathscr{R}(A,B) = \mathrm{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}. \tag{54}$$

As before, x_2 and x_3 do not affect the observations, but now only x_1 and x_3 are controllable. Thus x_3 can be immediately ignored, but in order to ignore x_2, it does not suffice to simply check that $\mathrm{Ker}(H) \subseteq \mathscr{R}(A,B)$ since x_2 is not controllable. Nonetheless, we do have $A\mathrm{Ker}(H) \subseteq \mathrm{Ker}(H) + \mathscr{R}(A,B)$, as the x_2 direction is A-invariant. We can therefore ignore both x_2 and x_3, and choose the same quotient map $y = Hx$ as in (51).

9. Conclusions

In this paper, we considered, characterized, and constructed bisimilar transition systems that were generated by discrete- and continuous-time linear control systems with finite observations. The characterizations, even though different for discrete- and continuous-time, relied on well-known control theoretic notions and algorithms.

The results are important as they pave the way for developing unified notions of bisimulation for discrete, continuous, and hybrid systems (Haghverdi, Tabuada, & Pappas, 2003). Simulations (not necessarily bisimulations) of transition systems generated by linear systems in the presence of state and input constraints will be critical in reducing the complexity of reachability computations that are used in the verification and design of hybrid systems (Asarin, Bournez, Dang, Maler, & Pnueli, 2000; Chutinam & Krogh, 2003; Tomlin, Lygeros, & Sastry, 2000). Finally, given the deep relationship between bisimulation and temporal logic, better understanding of bisimulation for continuous systems will be fundamental in designing controllers for continuous or hybrid systems, but with respect to temporal logic specifications (Tabuada & Pappas, 2003).

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