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Discrete Abstraction of Stochastic Nonlinear Systems: A Bisimulation Function Approach

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Abstract— **This paper addresses the discrete abstraction problem for stochastic nonlinear systems with continuous-valued state. The proposed solution is based on a function, called the bisimulation function, which provides a sufficient condition for the existence of a discrete abstraction for a given continuous system. We first introduce the bisimulation function and show how the function solves the problem. Next, a convex optimization based method for constructing a bisimulation function is presented. Finally, the proposed framework is demonstrated by a numerical simulation.**

I. Introduction

System abstraction, i.e., extracting a simpler but qualitatively similar model from a given system, has recently aroused great interest. The reason lies in its great potential for analysis and control of highly complex systems. For example, when one wants to verify if a system satisfies a certain property, the use of its abstracted model drastically reduces the computational complexity.

For this topic, various results have been extensively obtained. For deterministic systems, system equivalence has been discussed based on the notion of bisimulation relation [1], [2], and its generalization, called the approximate bisimulation, has been proposed in [3]. Moreover, for stochastic systems, the bisimulation notion has been developed in [4]–[6]. These works have provided fundamental theories of system abstraction. More concrete methodologies to abstract systems have been studied in [7]–[18]. They can be classified as Table I, where the systems with continuous-valued state and those with discrete-valued state are respectively called the *continuous systems* and the *discrete systems*. Item (i) corresponds to the reduction of a continuous system to a continuous system with lower dimensional state space, while (ii) is the reduction of a continuous system to a finite-state machine, which is called the *discrete abstraction*. On the other hand, (a) and (b) are distinguished by whether the original systems are deterministic or stochastic.

Here, we are interested in a problem in (ii)-(b), i.e., the discrete abstraction of stochastic systems. This is motivated by the recent result [19] on the biological control. There, the stochastic continuous system model of a biological system is abstracted into a Markov chain with two discrete states. Then, by exploiting good properties of the Markov chain, it has succeeded in establishing a promising control framework. However, the abstracted model is derived by the Monte Carlo method with a large number of numerical simulations. So we need to develop a more systematic method to abstract stochastic continuous systems to Markov chains. In addition, it should be noticed that, as shown in Table I, a discrete abstraction technique for stochastic systems has been proposed in [17], [18]. However, the resulting systems are *not* the standard Markov chains but the Markov *set*-chains which are more challenging to utilize than the standard ones.

TABLE I
Results on system abstraction.

		(a) deterministic	(b) stochastic
(i)	continuous system to continuous system	[7]–[9]	[10]–[15]
(ii)	continuous system to discrete system	[16]	[17], [18], [This paper]

This paper thus addresses the discrete abstraction of stochastic nonlinear systems to Markov chains, shown in Fig. 1. This abstraction reduces analysis and control problems for continuous systems into those for Markov chains, to which the existing useful techniques can be applied. For example, a basic issue for stochastic systems is the so-called reachability problem, that is, to compute the probability that the system does not reach an undesirable state set. For continuous systems, the problem is in general difficult to solve due to its exponential complexity with the state dimension. In contrast, it can be easily solved for Markov chains, because, as is well known, various probabilities on the system evaluation can be easily computed from the stochastic state transition matrices.

In this paper, to solve the discrete abstraction problem, we introduce a function, called the bisimulation function, which provides a sufficient condition for the existence of a Markov chain which is bisimilar to a given original system. Although the bisimulation function has been originally proposed in [3], the function proposed here is slightly different; the original is analysis-oriented, while ours is rather design-oriented. After introducing the bisimulation function, we next propose a method for deriving a bisimulation function. This is based on convex optimization, which enables efficient computation. Finally, the proposed framework is demonstrated by a numerical simulation.

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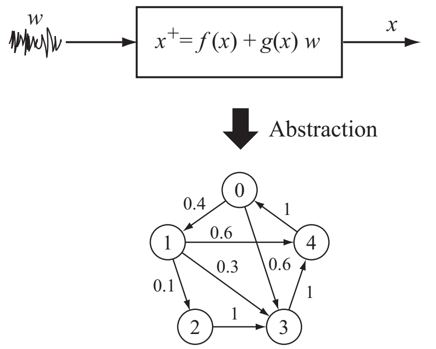

Fig. 1. Discrete Abstraction of Stochastic Systems into Markov Chains.

Notation: Let $\mathbf{R}$, $\mathbf{R}_{0+}$, and $\mathbf{N}$ be the real number field, the set of nonnegative real numbers, and the set of positive integers, respectively. We denote by $\mathbf{P}^{n\times n}$ the set of $n \times n$ stochastic matrices, and denote by $\mathbf{B}(x, \varepsilon)$ the closed ball of center x and radius ε. We use I_n to express the $n\times n$ identity matrix, and $M_1 \otimes M_2$ to express the Kronecker product of the matrices M_1 and M_2. For the random variable w, let $E[w]$ be the expected value and let $E[w|\pi]$ be the expected value when the event π occurs. Finally, for the vector x and the matrix M, the symbols $\|x\|$ and $\|M\|$ express the Euclidean norm and the Frobenius norm, respectively, i.e., $\|x\| = \sqrt{x^\top x}$ and $\|M\| = \sqrt{\mathrm{tr}(M^\top M)}$.

II. Problem Formulation

Consider the discrete-time nonlinear system

$$\Sigma_c: \ x(t+1) = f(x(t)) + g(x(t))w(t) \tag{1}$$

where $x \in \mathbf{R}^n$ is the state, $w \in \mathbf{R}^m$ is the stochastic process, and $f : \mathbf{R}^n \to \mathbf{R}^n$ and $g : \mathbf{R}^n \to \mathbf{R}^{n\times m}$ are functions. The initial state is given as $x(0) \in \mathbf{X}_0$ for a bounded set $\mathbf{X}_0 \subset \mathbf{R}^n$. For the process w, it is assumed that

- (A1) $w(t) \in \mathbf{W}$ for a bounded set $\mathbf{W} \subset \mathbf{R}^m$,
- (A2) $E[w(t)|x(t) = \xi] = 0$ for all $\xi \in \mathbf{R}^n$,
- (A3) $E[w(t)w^\top(t)|x(t) = \xi] = W(\xi)$ for a given variance-covariance matrix $W(\xi) \in \mathbf{R}^{m\times m}$ (which depends on $x(t)$).

The first assumption means that w is bounded. The second and third ones specify the expected value and the variance. Note here that (A2) does not lose any generality; when $E[w(t)|x(t) = \xi] = e(\xi) \neq 0$, we recover the same results for the system transformed with the new input valuable $\bar{w}(t) := w(t) - e(x(t))$.

In this paper, we are interested in abstracting Σ_c into the following Markov chain:

$$\Sigma_d(P): \ \Pr[\, z(t+1) = \zeta_j \,|\, z(t) = \zeta_i \,] = P_{ij} \tag{2}$$

where $z \in \{\zeta_1, \zeta_2, \ldots, \zeta_N\}$ ($\zeta_i \in \mathbf{R}^n$) is the state, which takes one of the N vector values, and $P_{ij} \in [0, 1]$ is the probability for the transition $\zeta_i \to \zeta_j$ in one time step. We express by P the stochastic state transition matrix, i.e., $P := [P_{ij}] \in \mathbf{P}^{N\times N}$.

The system Σ_c and its state are often called the *continuous system* and the *continuous state*, respectively. Likewise, the system $\Sigma_d(P)$ and its state are called the *discrete system* and the *discrete state*. In addition, the reachable set of Σ_c is defined as

$$\mathrm{Reach}(\Sigma_c) := \left\{ x^+ \in \mathbf{R}^n \,\middle|\, \begin{array}{l} \exists (t, x_0, w_0, \ldots, w_{t-1}) \in \mathbf{N} \times \mathbf{X}_0 \times \mathbf{W}^t \\ \text{s.t.} \ \ x^+ = x(t, x_0, w_0, \ldots, w_{t-1}) \end{array} \right\}$$

where $x(t, x_0, w_0, \ldots, w_{t-1})$ is the state $x(t)$ under the condition $x(0) = x_0$, $w(0) = w_0$, $\ldots$, $w(t-1) = w_{t-1}$.

For evaluating the distance between the two systems Σ_c and $\Sigma_d(P)$, we employ

$$\Delta_1(\xi, \zeta_i, P) := \big\| \, E[\, x(t{+}1) \,|\, x(t){=}\xi \,] - E[\, z(t{+}1) \,|\, z(t){=}\zeta_i \,] \, \big\|, \tag{3}$$

$$\Delta_2(\xi, \zeta_i, P) := \big\| \, E[\, x(t{+}1)\, x^\top(t{+}1) \,|\, x(t) = \xi \,] - E[\, z(t{+}1)\, z^\top(t{+}1) \,|\, z(t) = \zeta_i \,] \, \big\|, \tag{4}$$

which are based on the first- and second-order moments of the states.

Definition 1 (ε-bisimulation): For the systems Σ_c and $\Sigma_d(P)$, suppose that a precision $\varepsilon \in \mathbf{R}_{0+}$ satisfying

$$\varepsilon \geq \max_{\xi \in \mathrm{Reach}(\Sigma_c)} \ \min_{i \in \{1,2,\ldots,N\}} \|\xi - \zeta_i\| \tag{5}$$

is given. Then the systems Σ_c and $\Sigma_d(P)$ are said to be ε-bisimilar (denoted by $\Sigma_c \simeq_\varepsilon \Sigma_d(P)$) if, for all (ξ, ζ_i) satisfying $\|\xi - \zeta_i\| \leq \varepsilon$, the relations

$$\Delta_1(\xi, \zeta_i, P) \leq \varepsilon, \tag{6}$$

$$\Delta_2(\xi, \zeta_i, P) \leq \varepsilon^2 \tag{7}$$

hold. ■

Note that (5) guarantees that, for each $x(t) \in \mathrm{Reach}(\Sigma_c)$, there exists a discrete state ζ_i which is an ε-neighbor of $x(t)$.

Note also that the right hand side of (7) is bounded by the square of ε, since Δ_2 is based on the second-order term of x and z. Other types of relations, such as $\Delta_1(\xi, \zeta_i, P) \leq \varepsilon$, $\Delta_2(\xi, \zeta_i, P) \leq \delta$ with independent values ε and δ, can be also handled by the straightforward extention.

Then the following problem is addressed in this paper.

Problem 1: For the continuous system Σ_c, suppose that the discrete states ζ_1, ζ_2, $\ldots$, ζ_N are given.
(i) Given a stochastic matrix $P \in \mathbf{R}^{N\times N}$ and a precision $\varepsilon \in \mathbf{R}_{0+}$, determine if $\Sigma_c \simeq_\varepsilon \Sigma_d(P)$.
(ii) Find a P and an ε satisfying $\Sigma_c \simeq_\varepsilon \Sigma_d(P)$. ■

Several remarks on Problem 1 are given.

First, the relation $\Sigma_c \simeq_\varepsilon \Sigma_d(P)$ allows us to easily (but approximately) solve the reachability problem for the continuous system Σ_c, that is, compute the probability that any initial state on a set does not reach an undesirable set. For example, the probability for Σ_c that any $x(0) \in \mathbf{X}_0 = \mathbf{B}(\zeta_1, \varepsilon)$ does not reach a set $\mathbf{X}_1 = \mathbf{B}(\zeta_N, \varepsilon)$ within time T is approximated by

$$\Pr[\, z(1) \neq \zeta_N, \ z(2) \neq \zeta_N, \ \ldots, \ z(T) \neq \zeta_N \,|\, z(0) = \zeta_1 \,]$$

for the discrete system $\Sigma_d(P)$. Then this is easily computed as

$$\prod_{t=1}^{T}(1-P_{1N}^t)$$

where P_{1N}^t is the $(1,N)$-th element of P^t (the stochastic transition matrix to the t-th power).

Second, in Problem 1, the discrete states $\{\zeta_1,\zeta_2,\ldots,\zeta_N\}$ are pre-fixed, though one might have a flexibility in choosing $\{\zeta_1,\zeta_2,\ldots,\zeta_N\}$ in some cases. This type of problem is considered in the situation where some key states in the dynamics are known a priori. For example, it has been pointed out in [19] that in a biological system, stable equilibria are dominant factors to describe the dynamics.

Finally, due to mathematical difficulty, it is too difficult to solve Problem 1 exactly. In fact, (i) and (ii) correspond to the so-called nonnegativity problem and the robust inequality problem, in which the arising inequalities ((6) and (7)) are nonconvex with respect to ξ (which will be shown later). This fact motivates us to introduce a *bisimulation function* which provides a sufficient condition for the existence of (P,ε) satisfying $\Sigma_c \simeq_\varepsilon \Sigma_d(P)$.

III. Bisimulation functions

A. Definition

In this paper, a bisimulation function is defined based on the decomposition of $\Delta_1(\xi,\zeta_i,P)$ and $\Delta_2(\xi,\zeta_i,P)$ in (3) and (4).

We decompose $\Delta_1(\xi,\zeta_i,P)$ into two parts:

$$\begin{aligned}&\Delta_1(\xi,\zeta_i,P)\\ &=\big\|E[x(t{+}1)\,|\,x(t){=}\,\xi]-E[x(t{+}1)\,|\,x(t){=}\,\zeta_i]\\ &\qquad+E[x(t{+}1)\,|\,x(t){=}\,\zeta_i]-E[\,z(t{+}1)\,|\,z(t)=\zeta_i\,]\big\|\\ &\le\big\|E[x(t{+}1)\,|\,x(t){=}\,\xi]-E[x(t{+}1)\,|\,x(t){=}\,\zeta_i]\big\|\\ &\quad+\big\|E[x(t{+}1)\,|\,x(t){=}\,\zeta_i]-E[\,z(t{+}1)\,|\,z(t)=\zeta_i\,]\big\| \qquad (8)\end{aligned}$$

Here, the first term expresses the difference by the state-space quantization and the second term does the difference of the dynamics. For simplicity of notation, we denote these terms by $\Delta_{11}(\xi,\zeta_i)$ and $\Delta_{12}(\zeta_i,P)$, i.e.,

$$\Delta_1(\xi,\zeta_i,P)\le\Delta_{11}(\xi,\zeta_i)+\Delta_{12}(\zeta_i,P). \qquad (9)$$

In a similar way to this, $\Delta_2(\xi,\zeta_i,P)$ can be decomposed as

$$\begin{aligned}\Delta_2(\xi,\zeta_i,P)\le&\;\big\|\,E[\,x(t{+}1)\,x^\top(t{+}1)|\,x(t){=}\,\xi\,]\\ &\qquad-E[\,x(t{+}1)\,x^\top(t{+}1)|\,x(t){=}\,\zeta_i\,]\,\big\|\\ &+\big\|\,E[\,x(t{+}1)\,x^\top(t{+}1)|\,x(t){=}\,\zeta_i\,]\\ &\qquad-E[\,z(t{+}1)\,z^\top(t{+}1)|\,z(t){=}\,\zeta_i\,]\,\big\|\\ =&\;\Delta_{21}(\xi,\zeta_i)+\Delta_{22}(\zeta_i,P) \qquad (10)\end{aligned}$$

where $\Delta_{21}(\xi,\zeta_i)$ and $\Delta_{22}(\zeta_i,P)$ are similarly defined. Note that $\Delta_{11}(\zeta_i,\zeta_i)=0$ and $\Delta_{21}(\zeta_i,\zeta_i)=0$. Then a bisimulation function is introduced as follows.

Definition 2 (Bisimulation functions): A function $\phi:\mathbf{R}_{0+}\times\{\zeta_1,\zeta_2,\ldots,\zeta_N\}\to\mathbf{R}$ is a *bisimulation function for* Σ_c *and* $\{\zeta_1,\zeta_2,\ldots,\zeta_N\}$ if
(a) $\phi(\|\xi-\zeta_i\|,\zeta_i)$ is differentiable with respect to $\|\xi-\zeta_i\|$,
(b) $\phi(\|\xi-\zeta_i\|,\zeta_i)\le\|\xi-\zeta_i\|-\Delta_{11}(\xi,\zeta_i)$,
(c) $\|\xi-\zeta_i\|\,\phi(\|\xi-\zeta_i\|,\zeta_i)\le\|\xi-\zeta_i\|^2-\Delta_{21}(\xi,\zeta_i)$,
(d) there exists a positive scalar ω such that

$$\omega\le\frac{\partial\phi(\|\xi-\zeta_i\|,\zeta_i)}{\partial\|\xi-\zeta_i\|},$$

(e) $\dfrac{\partial\phi(\|\xi-\zeta_i\|,\zeta_i)}{\partial\|\xi-\zeta_i\|}\le 1$. ∎

B. Significance of Bisimulation Functions

The significance of the bisimulation function is stated as follows.

Theorem 1: If there exists a bisimulation function ϕ for Σ_c and $\{\zeta_1,\zeta_2,\ldots,\zeta_N\}$, the following statements hold.
(i) There exist a stochastic matrix P and a precision ε such that $\Sigma_c\simeq_\varepsilon\Sigma_d(P)$.
(ii) If the pair (P,ε) satisfies

$$-\phi(\varepsilon,\zeta_i)+\Delta_{12}(\zeta_i,P)\le 0 \quad (i=1,2,\ldots,N), \qquad (11)$$
$$-\varepsilon\phi(\varepsilon,\zeta_i)+\Delta_{22}(\zeta_i,P)\le 0 \quad (i=1,2,\ldots,N), \qquad (12)$$

then $\Sigma_c\simeq_\varepsilon\Sigma_d(P)$.

Proof: First, we prove (ii). By applying Def. 2 (b) and (11) to (9), it follows that

$$\Delta_1(\xi,\zeta_i,P)\le\|\xi-\zeta_i\|-\phi(\|\xi-\zeta_i\|,\zeta_i)+\phi(\varepsilon,\zeta_i)$$

holds for every $(\xi,\zeta_i)\in\mathbf{R}^n\times\{\zeta_1,\zeta_2,\ldots,\zeta_N\}$. From Def. 2 (e), $\|\xi-\zeta_i\|-\phi(\|\xi-\zeta_i\|,\zeta_i)$ is monotonically nondecreasing with $\|\xi-\zeta_i\|$, which implies that if $\|\xi-\zeta_i\|\le\varepsilon$, then

$$\Delta_1(\xi,\zeta_i,P)\;\le\;\varepsilon-\phi(\varepsilon,\zeta_i)+\phi(\varepsilon,\zeta_i)\;\le\;\varepsilon.$$

On the other hand, in a similar way to the above, it can be shown from (10), (12), and Def. 2 (c) that

$$\begin{aligned}&\Delta_2(\xi,\zeta_i,P)\\ &\le\|\xi-\zeta_i\|^2-\|\xi-\zeta_i\|\;\phi(\|\xi-\zeta_i\|,\zeta_i)+\varepsilon\phi(\varepsilon,\zeta_i).\end{aligned}$$

Then since Def. 2 (b) and (e) imply that $\|\xi-\zeta_i\|(\|\xi-\zeta_i\|-\phi(\|\xi-\zeta_i\|,\zeta_i))$ is monotonically nondecreasing with $\|\xi-\zeta_i\|$, we have

$$\Delta_2(\xi,\zeta_i,P)\;\le\;\varepsilon^2-\varepsilon\phi(\varepsilon,\zeta_i)+\varepsilon\phi(\varepsilon,\zeta_i)\;\le\;\varepsilon^2$$

under the condition $\|\xi-\zeta_i\|\le\varepsilon$. These mean $\Sigma_c\simeq_\varepsilon\Sigma_d(P)$. Next, (i) is proven. Def. 2 (d) means that $-\phi(\|\xi-\zeta_i\|,\zeta_i)$ is monotonically decreasing with $\|\xi-\zeta_i\|$. Furthermore, it means that $-\|\xi-\zeta_i\|\phi(\|\xi-\zeta_i\|,\zeta_i)$ is monotonically decreasing with $\|\xi-\zeta_i\|$ on $[\bar{\varepsilon},\infty)$ ($\bar{\varepsilon}$ is some value). So for any stochastic matrix P, there exists an ε satisfying (11) and (12). This and (ii) imply (i). ∎

Statement (i) provides a sufficient condition for the continuous system Σ_c to have an ε-bisimilar system $\Sigma_d(P)$, and (ii) characterizes (P,ε) for the bisimulation by $2N$ inequalities. Note that (i) does not always hold, because, for example, an unstable continuous system cannot be approximated by any finite state machine.

Once a bisimulation function is obtained, the solutions to Problem 1 can be readily derived. The decision problem (i) is solved by checking the satisfaction of (11) and (12) for the given (P,ε). On the other hand, (ii) is resolved by finding a pair (P,ε) satisfying (11) and (12). For example, a solution with the minimum ε, which may be the most useful, is given as follows.

Theorem 2: The pair $(P(\infty),\varepsilon(\infty))$ given by the following algorithm is an asymptotic solution of (11) and (12) with the minimum ε.

(Algorithm BISIM)

(Step 1) Set

$$\begin{aligned}&\varepsilon_{min} := \text{the minimum } \varepsilon \text{ satisfying (5)},\\ &\varepsilon_{max} := \text{a sufficiently large positive number},\\ &\varepsilon(0) := \frac{\varepsilon_{min}+\varepsilon_{max}}{2},\\ &i := 0 \quad \text{(counter initialization).}\end{aligned}$$

(Step 2) Solve the following optimization problem and let $(\gamma(i),P(i))$ be the solution.

$$\min_{\gamma\in\mathbf{R},\, P\in\mathbf{P}^{N\times N}} \quad \gamma$$
$$\text{s.t.}\begin{cases} -\phi(\varepsilon(i),\zeta_i) + \big\| f(\zeta_i) - [\zeta_0\ \zeta_1 \cdots \zeta_N]P^\top e_i \big\| \le \gamma \\ \qquad\qquad (i=1,2,\dots,N), \\ -\varepsilon(i)\phi(\varepsilon(i),\zeta_i) + \big\| f(\zeta_i)f^\top(\zeta_i) + g(\zeta_i)W(\zeta_i)g^\top(\zeta_i) \\ \quad - [\zeta_0\zeta_0^\top\ \ \zeta_1\zeta_1^\top\ \cdots\ \zeta_N\zeta_N^\top](P^\top \otimes I_{nN})E_i \big\| \le \gamma \\ \qquad\qquad (i=1,2,\dots,N)\end{cases}$$

where $e_i := [0\cdots 0\ 1\ 0\ \cdots 0]^\top$ is the i-th standard basis in $\mathbf{R}^N$ and E_i is the $nN\times n$ matrix of the form

$$E_i := [0_{n\times n}\ \cdots\ 0_{n\times n}\ \underset{\substack{\uparrow\\ \text{the } i\text{-th block}}}{I_n}\ 0_{n\times n}\ \cdots\ 0_{n\times n}]^\top.$$

(Step 3) If $\gamma(i)>0$,

$$\varepsilon(i+1) := \frac{\varepsilon_{max}+\varepsilon(i)}{2}, \quad \varepsilon_{min} := \varepsilon(i),$$

otherwise

$$\varepsilon(i+1) := \frac{\varepsilon_{min}+\varepsilon(i)}{2}, \quad \varepsilon_{max} := \varepsilon(i).$$

(Step 4) $i := i+1$ and go to Step 2.

Proof: See Appendix I. ■

The above method is based on the convex optimization for P (and γ) and the bisection search for ε. Thus a solution to Problem 1 (ii) can be efficiently computed.

As a consequence of the above discussion, Problem 1 can be reduced into finding a bisimulation function ϕ. In the next section, we propose a computationally tractable method to derive a bisimulation function.

IV. Construction of Bisimulation Functions

In order to compute bisimulation functions, the following result plays an important role.

Theorem 3: All bisimulation function for Σ_c and $\{\zeta_1,\zeta_2,\dots,\zeta_N\}$ are given by

$$\phi(\|\xi-\zeta_i\|,\zeta_i) = \|\xi-\zeta_i\| - \sqrt{\alpha(\|\xi-\zeta_i\|,\zeta_i)} \qquad (13)$$

where $\alpha:\mathbf{R}_{0+}\times\{\zeta_1,\zeta_2,\dots,\zeta_N\}\to\mathbf{R}_{0+}$ is the parameter function satisfying

(a') $\alpha(\|\xi-\zeta_i\|,\zeta_i)$ is differentiable with respect to $\|\xi-\zeta_i\|$,

(b') $\alpha(\|\xi-\zeta_i\|,\zeta_i) - \|f(\xi)-f(\zeta_i)\|^2 \ge 0$,

(c') $\alpha(\|\xi-\zeta_i\|,\zeta_i)\|\xi-\zeta_i\|^2$
$$- \|f(\xi)f^\top(\xi) + g(\xi)W(\xi)g^\top(\xi) - f(\zeta_i)f^\top(\zeta_i) - g(\zeta_i)W(\zeta_i)g^\top(\zeta_i)\|^2 \ge 0,$$

(d') there exists a positive scalar ω such that
$$4(1-\omega)^2\alpha(\|\xi-\zeta_i\|,\zeta_i) - \left(\frac{\partial\alpha(\|\xi-\zeta_i\|,\zeta_i)}{\partial\|\xi-\zeta_i\|}\right)^2 \ge 0,$$

(e') $\dfrac{\partial\alpha(\|\xi-\zeta_i\|,\zeta_i)}{\partial\|\xi-\zeta_i\|} \ge 0$,

where f, g, and W are defined in Section II.

Proof: It is obvious that (13) is a bijective relation between ϕ and α (note $\alpha(\|\xi-\zeta_i\|,\zeta_i)\in\mathbf{R}_{0+}$). So we show that (a)–(e) are equivalent to (a')–(e').

(a) ↔ (a'): Trivial from (13).

(b) ↔ (b'): By calculating an explicite form of $E[x(t+1)\,|\,x(t)=\xi]$, we have

$$\Delta_{11}(\xi,\zeta_i) = \|f(\xi)-f(\zeta_i)\|. \qquad (14)$$

This and (13) prove that (b) and (b') are equivalent.

(c) ↔ (c'): By a straightforward calculation, we can derive

$$\Delta_{21}(\xi,\zeta_i) = \|f(\xi)f^\top(\xi) + g(\xi)W(\xi)g^\top(\xi) - f(\zeta_i)f^\top(\zeta_i) - g(\zeta_i)W(\zeta_i)g^\top(\zeta_i)\|. \quad (15)$$

This and (13) imply that (c) and (c') are equivalent.

(d) ↔ (d'): Consider the inequality in (d'). By transposing the second term to the right hand side and taking the square root of both sides, the inequality is expressed as

$$2(1-\omega)\sqrt{\alpha(\|\xi-\zeta_i\|,\zeta_i)} \ge \frac{\partial(\sqrt{\alpha(\|\xi-\zeta_i\|,\zeta_i)})^2}{\partial\|\xi-\zeta_i\|},$$

and, equivalently,

$$2(1-\omega)\sqrt{\alpha(\|\xi-\zeta_i\|,\zeta_i)} \ge 2\sqrt{\alpha(\|\xi-\zeta_i\|,\zeta_i)}\frac{\partial\sqrt{\alpha(\|\xi-\zeta_i\|,\zeta_i)}}{\partial\|\xi-\zeta_i\|}.$$

Furthermore, this is represented as

$$1 - \frac{\partial\sqrt{\alpha(\|\xi-\zeta_i\|,\zeta_i)}}{\partial\|\xi-\zeta_i\|} \ge \omega.$$

Since (13) implies

$$\frac{\partial\phi(\|\xi-\zeta_i\|,\zeta_i)}{\partial\|\xi-\zeta_i\|} = 1 - \frac{\partial\sqrt{\alpha(\|\xi-\zeta_i\|,\zeta_i)}}{\partial\|\xi-\zeta_i\|}, \qquad (16)$$

it is shown that (d') is equivalent to (d).

(e) $\leftrightarrow$ (e'): Trivial from (13). ■

In Definition 2, the bisimulation function ϕ is introduced with the properties (b) and (c) including square-root terms, e.g., $\|\xi-\zeta_i\|$ $(=\sqrt{(\xi-\zeta_i)^\top(\xi-\zeta_i)})$. On the other hand, Theorem 3 provides a parameterization of ϕ by the function α which is not characterized by square-root terms (in (b') and (c')). This enables us to derive a bisimulation function via a sum of squares problem (which is convex! [20]).

Suppose that the elements of f, g, and W are quotients of a polynomial by a positive polynomial (or can be approximated by them), and α is of the form

$$\alpha(\|\xi-\zeta_i\|,\zeta_i) = b_i\|\zeta_i\|^2 + \sum_{j=1}^{M} c_{ij}\|\xi-\zeta_i\|^{2j} \tag{17}$$

where $b_i, c_{ij} \in \mathbf{R}$ are coefficients and $M \in \mathbf{N}$ is an accuracy parameter selected by users. Then the function α is constructed with the solution to the following sum of squares problem.

$$\begin{aligned}&\text{Find} \quad b_i,\ c_{ij} \quad (i=1,2,\ldots,N,\ j=1,2,\ldots,M)\\ &\text{s.t.} \begin{cases}\text{Left hand side of (b')} \times p_1(\xi) \text{ is a sum of squares,}\\ \text{Left hand side of (c')} \times p_2(\xi) \text{ is a sum of squares,}\\ \text{Left hand side of (d') is a sum of squares,}\\ \text{Left hand side of (e')} \times \|\xi-\zeta_i\| \text{ is a sum of squares,}\\ \hfill (i=1,2,\ldots,N),\end{cases}\end{aligned}$$

where p_1, p_2 and ω are arbitrarily given positive polynomials and a small positive scalar. Note that α and

$$\left(\frac{\partial\alpha(\|\xi-\zeta_i\|,\zeta_i)}{\partial\|\xi-\zeta_i\|}\right)^2$$

are polynomials of $\|\xi-\zeta_i\|^2$ $(=(\xi-\zeta_i)^\top(\xi-\zeta_i))$, i.e., polynomials of ξ. In addition, notice that, since (b'), (c'), and (e') are not always polynomial conditions, they are transformed into equivalent polynomial conditions by introducing the positive polynomials p_1, p_2 and the positive term $\|\xi-\zeta_i\|$.

By solving this sum of squares problem, we can derive a function α and thus obtain a bisimulation function ϕ.

V. Example

Consider the following continuous system

$$\Sigma_c : \begin{cases} x_1(t+1) = 0.3x_1(t) + \dfrac{x_2(t)}{3+x_1^2(t)x_2^2(t)},\\ x_2(t+1) = -0.15x_1(t) + 0.3x_2(t) + w(t)\end{cases}$$

where $x_i \in \mathbf{R}$ $(i\in\{1,2\})$, $w \in \mathbf{W} := [-1,1]$, $E(w)=0$, and $E(w^2)=0.3$. The discrete states of $\Sigma_d(P)$ are given by

$$\zeta_1 := \begin{bmatrix}-1\\-1\end{bmatrix},\quad \zeta_2 := \begin{bmatrix}-1\\0\end{bmatrix},\quad \zeta_3 := \begin{bmatrix}-1\\1\end{bmatrix},$$
$$\zeta_4 := \begin{bmatrix}-1\\0\end{bmatrix},\quad \zeta_5 := \begin{bmatrix}0\\0\end{bmatrix},\quad \zeta_6 := \begin{bmatrix}0\\1\end{bmatrix},$$

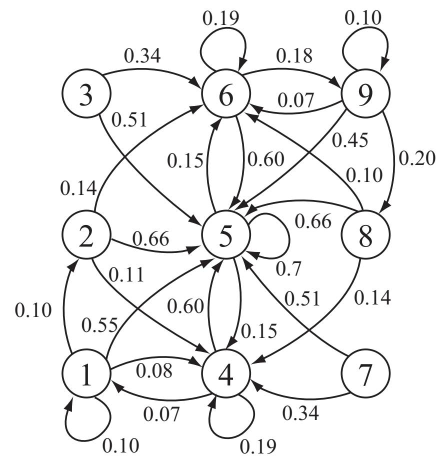

Fig. 2. Discrete abstraction by the proposed method. In this figure, some nodes with small probability are omitted.

$$\zeta_7 := \begin{bmatrix}1\\-1\end{bmatrix},\quad \zeta_8 := \begin{bmatrix}1\\0\end{bmatrix},\quad \zeta_9 := \begin{bmatrix}1\\1\end{bmatrix}.$$

Then the proposed method provides the bisimulation function

$$\phi(\|\xi-\zeta_i\|,\zeta_i) = \|\xi-\zeta_i\| - \sqrt{0.27\|\xi-\zeta_i\|^2 + 0.26\|\zeta_i\|^2}.$$

This guarantees that there exists a stochastic matrix P and a precision ε such that $\Sigma_c \simeq_\varepsilon \Sigma_d(P)$ (see Theorem 1). Using this, we obtain the Markov chain $\Sigma_d(P)$ in Fig. 2 with $\varepsilon \simeq 0.8$, where some nodes with small probability are omitted.

For the system Σ_c and the discrete states $\{\zeta_1,\zeta_2,\ldots,\zeta_9\}$, the minimum ε satisfying (5) is $\sqrt{2}/2$ $(\simeq 0.707)$. Compared with this, it turns out that the discrete system $\Sigma_d(P)$ with $\varepsilon \simeq 0.8$ is a good approximation of Σ_c.

In this way, the proposed method solves the discrete abstraction problem for stochastic continuous systems.

VI. Conclusion

This paper has considered the discrete abstraction problem for stochastic nonlinear systems. The problem has been reduced into the problem of finding the bisimulation function, which provides a systematic method to abstract stochastic continuous systems to Markov chains. We also have presented a construction technique of the bisimulation function based on sum of squares programming.

Appendix I
Proof of Theorem 2

A. Preliminary: Explicite Formulas of $\Delta_{12}(\zeta_i,P)$ and $\Delta_{22}(\zeta_i,P)$

By a straightforward calculation with Assumptions (A2), (A3) and equations (3), (4), we have the following explicite formulas of $\Delta_{12}(\zeta_i, P)$ and $\Delta_{22}(\zeta_i, P)$:

$$\Delta_{12}(\zeta_i, P) = \left\| f(\zeta_i) - [\zeta_0\ \zeta_1 \cdots \zeta_N] P^\top e_i \right\|, \tag{18}$$

$$\Delta_{22}(\zeta_i, P) = \left\| f(\zeta_i) f^\top(\zeta_i) + g(\zeta_i) W(\zeta_i) g^\top(\zeta_i) - [\zeta_0\zeta_0^\top\ \ \zeta_1\zeta_1^\top \cdots \zeta_N\zeta_N^\top](P^\top \otimes I_{nN}) E_i \right\|. \tag{19}$$

B. Proof of Main Part

First, Algorithm BISIM corresponds to the (standard) bisection root finding method for the following scalar equation with the variable ε:

$$\min_{P\in\mathbf{P}^{N\times N}} \max_{i\in\{1,2,\ldots,N\}} \max\{-\phi(\varepsilon,\zeta_i) + \Delta_{12}(\zeta_i, P), -\varepsilon\phi(\varepsilon,\zeta_i) + \Delta_{22}(\zeta_i, P)\} = 0. \tag{20}$$

Thus we have a solution to (20) by the procedure.

Next, we show that (20) holds for the solution of (11) and (12) with the minimum ε. From (3) and (4), $\Delta_{12}(\zeta_i, P) \geq 0$ and $\Delta_{22}(\zeta_i, P) \geq 0$. Thus it follows that

$$\phi(\varepsilon, \zeta_i) \geq 0 \tag{21}$$

holds for the all solutions to (11) and (12). For ε satisfying (21), $-\phi(\varepsilon,\zeta_i)$ and $-\varepsilon\phi(\epsilon,\zeta_i)$ are monotonically decreasing, which means that ε is the minimum if one of the $2N$ terms

$$\min_{P\in\mathbf{P}^{N\times N}} -\phi(\varepsilon,\zeta_i) + \Delta_{12}(\zeta_i, P) \quad (i = 1, 2, \ldots, N),$$
$$\min_{P\in\mathbf{P}^{N\times N}} -\varepsilon\phi(\varepsilon,\zeta_i) + \Delta_{22}(\zeta_i, P) \quad (i = 1, 2, \ldots, N)$$

is zero and the others are nonpositive. This condition is expressed in (20), which completes the proof.

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