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Sparsest Minimum Multiple-Cost Structural Leader Selection ⋆

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Abstract: In this paper, we address the problem of leader selection in a network of agents when the input signal can originate from different sources (for instance communication/control towers) and incur different costs. We assume that each agent's state is defined by a scalar that evolves according to linear dynamics involving the states of its neighbors and its own state. We propose an algorithm to determine the minimum number of agents that should behave as leaders (i.e., agents whose states serve as references (or inputs) to the remaining agents–the followers), while incurring the lowest cost. Finally, illustrative examples using the main results of the paper are provided.

Keywords: Graph-Based Methods for Networked Systems, Multi-Agent Systems, Leader-Selection, Structural Controllability

The advent of micro and nano-scale sensors and actuators has led to the possibility of controlling larger and more complex networks. A crucial question is often how to optimally control such networks. For example, what is the minimum number of states which must be actuated to guarantee system controllability? How should inputs be assigned when controlling different states at different costs? In addition, the parametric realization of the system's dynamics may not be accurately known, in which case structural systems theory (Dion et al. (2003)) provides the tools to practically answer such questions.

In this work, we explain how to optimally select inputs to ensure structural controllability of a specific type of networked dynamical system: multi-agent networks. More precisely, we consider scenarios where signals are provided to the different *agents* by multiple communication/control *towers*. Each tower can provide individual control signals, which we refer to as *dedicated inputs*, to multiple agents with a cost which depends on the agents' locations or characterizations. In particular, the cost can be related to the distance between the tower and agent, or preferential assignment in a heterogenous multi-agent network, where some towers can only provide signals to agents of a specified type.

The problem addressed hereafter is closely related to the so-called (unconstrained) *leader selection problem* (Pequito et al. (2014c)), where agents to whom inputs are assigned are deemed *leaders* in the network. Therefore, the goal is to determine the minimum number of such leaders that ensure that controllability of the multi-agent network is ensured. Nonetheless, we add a new level of complexity to the problem by assuming that these signals can originate from different actuators, for instance, communication/control towers. In addition, due to uncertainty or unknown parameters of the dynamics we consider the structural counterpart of the problem; hence, we aim to ensure structural controllability, see Section 1 for formal definition and statement of the problem.

The leader selection problem is often approached as a minimization of the energy cost (Tzoumas et al. (2014)), number of leaders (Pequito et al. (2013c); Pequito et al. (2014b); Jiang et al. (2009); Commault and Dion (2013); Pequito et al. (2014c)), assignability cost (Pequito et al. (2013b,a)), network coherence (Patterson and Bamieh (2010)), mean square error with respect to the reference trajectory, or variants of the former (Lin et al. (2014); Clark and Poovendran (2011); Clark et al. (2014); Patterson et al. (2014); Patterson (2014); Shames et al. (2010); Franchi et al. (2011)). At last, as we previously pointed out, the leader selection problems can be seen as a particular case of input selection problems, with similar optimization objectives (Pasqualetti et al. (2014); Tzoumas et al. (2014); Summers et al. (2014); Olshevsky (2014); Pequito et al. (2014c,b); Ramos et al. (2014)). Subsequently, the closest paper to the work presented hereafter is (Pequito et al. (2014a)), where the problem of determining the minimum collection of dedicated inputs incurring the lowest cost is analyzed. In particular, that work deals with the problem explored by us in this paper when only one communication tower is considered. Therefore, our results are more general, and can be applied to the two scenarios above described, to which the results in (Pequito et al. (2014a)) can only provide insights.

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The main contributions of this paper are as follows: we extend the sparsest minimum cost leader selection problem to the case where an entity can provide input signals to different leaders, i.e., the inputs are multiple dedicated, while incurring different costs.

The rest of the paper is organized as follows. In Section 1, we provide the formal problem statement. Section 2 reviews some concepts and introduces results in structural systems theory. Subsequently, in Section 3, we present the main technical results, followed by an illustrative example in Section 4. Conclusions and discussions on further research are presented in Section 5.

1. PROBLEM STATEMENT

Let $\mathcal{D} = (\mathcal{V}, \mathcal{E})$ denote a communication graph connecting a set of agents, represented by the vertices in $\mathcal{V}$, and a set of directed edges $\mathcal{E}$. A directed edge $(i, j) \in \mathcal{E}$ indicates that agent i is able to transmit information to agent j. We denote by $\mathcal{N}_i^-$ the in-neighbors of agent i, i.e., all the agents $j \neq i$ such that $(j, i) \in \mathcal{E}$. Further, an agent can communicate with itself, but may also work as a simple integrator. Subsequently, agents are assumed to possess a scalar state, and agent i updates its state according to the following linear update:

$$x_i[k+1] = a_{ii}x_i[k] + \sum_{j\in\mathcal{N}_i^-} a_{ij}x_j[k], \qquad (1)$$

where $x_i[k]$ is the state of agent i at time k, and $\{a_{ij} : j \in \mathcal{N}_i^- \cup \{i\}\}$ is the set of weights that determines the protocol run by agent i. The multi-agent network dynamics can be re-written as a discrete-time linear time-invariant dynamical system:

$$x[k+1] = A(\mathcal{D})x[k], \qquad (2)$$

where $A(\mathcal{D})$ is the dynamics induced by the communication graph $\mathcal{D}$, with $[A(\mathcal{D})]_{i,j} = 0$ if $(j, i) \notin \mathcal{E}$. Whereas (2) represents an autonomous dynamical system, we are interested in the case where some of the agents are driven by exogenous input signals provided by the communication/control towers. We call these agents 'leaders' and describe the resulting non-autonomous dynamics by:

$$x_i[k+1] = a_{ii}x_i[k] + \sum_{j\in\mathcal{N}_i^-} a_{ij}x_j[k] + b_{il}u_i[k], \qquad (3)$$

where $b_{il} = 1$ if agent i is a leader, and $b_{il} = 0$ otherwise; in addition, the index l indicates the actuator that provided the signal to the leader, for instance, the communication/control tower. Subsequently, we can rewrite (3) in matrix form as

$$\dot{x}(t) = Ax(t) + \underbrace{[\mathbb{I}_n(\mathcal{J}_1)\ \ldots\ \mathbb{I}_n(\mathcal{J}_p)]}_{\mathbb{I}_n(\mathcal{J}_1,\ldots,\mathcal{J}_p)} u(t), \qquad (4)$$

where $\mathbb{I}_n(\mathcal{J})$ corresponds to the collection of columns of the $n \times n$ identity matrix with index in $\mathcal{J}$, where $\mathcal{J}$ is the set of indices corresponding to the leaders. In addition, $x \in \mathbb{R}^n$ denotes the collection of the agents' states, and $u \in \mathbb{R}^{|\mathcal{J}_1|+\ldots+|\mathcal{J}_p|}$, with $\mathcal{J}_i \subset \{1, \ldots, n\}$, is the input. The input is given by $u(t) = [u_1(t)\ \ldots\ u_p(t)]$, where $u_i \in \mathbb{R}^{|\mathcal{J}_i|}$ corresponds to the signals provided by the different communication/control towers deployed in a certain geographical area, and the communication towers can emit signals to the different agents in the surrounding area.

In the sequel, we identify the system in (4) with the tuple $(A, \mathbb{I}_n(\mathcal{J}_1, \ldots, \mathcal{J}_p))$. In many practical real scenarios with particular emphasis of large-scale systems, it is often the case that the exact values of the non-zero parameters of the plant matrices are unknown, or that these may change over time. To circumvent this problem, in this paper we adopt the the framework of structural systems (Dion et al. (2003)). To this end, we let $\bar{A} \in \{0, 1\}^{n\times n}$ and $\mathbb{I}_n(\mathcal{J}_1, \ldots, \mathcal{J}_p) \in \{0, 1\}^{n\times(|\mathcal{J}_1|+\ldots+|\mathcal{J}_p|)}$ be the binary matrices that represent the structural patterns (location of zeros and non-zeros) of A and $\mathbb{I}_n(\mathcal{J}_1, \ldots, \mathcal{J}_p)$, respectively. Then, we then focus on properties of systems, where the plant matrices have these sparsity patterns $(\bar{A}, \mathbb{I}_n(\mathcal{J}_1, \ldots, \mathcal{J}_p))$ which we refer to as a *structural system*. A pair $(\bar{A}, \mathbb{I}_n(\mathcal{J}_1, \ldots, \mathcal{J}_p))$ is said to be structurally controllable if there exists a pair $(A', \mathbb{I}_n(\mathcal{J}_1, \ldots, \mathcal{J}_p))$ with the same structure as $(\bar{A}, \mathbb{I}_n(\mathcal{J}_1, \ldots, \mathcal{J}_p))$, i.e., with the same locations of zeros and nonzeros, such that $(A', \mathbb{I}_n(\mathcal{J}_1, \ldots, \mathcal{J}_p))$ is controllable. By density arguments (Reinschke (1988)), it can be shown that if a pair $(\bar{A}, \mathbb{I}_n(\mathcal{J}_1, \ldots, \mathcal{J}_p))$ is structurally controllable, then almost all (with respect to the Lebesgue measure) pairs with the same structure as $(\bar{A}, \mathbb{I}_n(\mathcal{J}_1, \ldots, \mathcal{J}_p))$ are controllable. In essence, structural controllability is a property of the structure of the pair $(\bar{A}, \mathbb{I}_n(\mathcal{J}_1, \ldots, \mathcal{J}_p))$ and not of the specific numerical values.

Further, the inputs can incur different costs, possibly related with the signal to noise ratio or signal strength that is proportional to the square of the distance between the communication tower and an agent. In summary, the problem addressed in this paper can be formally stated as follows:

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$\mathcal{P}_1$: Let $\bar{A} \in \{0, 1\}^{n\times n}$ and $\mathbb{I}_n(\mathcal{J}_1, \ldots, \mathcal{J}_p) \in \{0, 1\}^{n\times|\mathcal{J}_1|+\ldots+|\mathcal{J}_p|}$ correspond to the dynamics and input matrices, respectively. In addition, let $c^i \in (\mathbb{R}_0^+ \cup \{\infty\})^{n\times n}$ be a vector where the entry c^i_j indicates the cost of actuating state variable j by actuator i. Then, we aim to determine $(\mathcal{J}_1^*, \ldots, \mathcal{J}_p^*)$, where $\mathcal{J}_i \subset \{1 \ldots, n\}$, is the solution to the following optimization problem:

$$\begin{aligned}\min_{\mathcal{J}_1,\ldots,\mathcal{J}_p} \quad & \sum_{k=1}^{p}\sum_{j\in\mathcal{J}_k} c^k_j \\ \text{s.t.} \quad & (\bar{A}, \mathbb{I}_n(\mathcal{J}_1, \ldots, \mathcal{J}_p)) \text{ is struct. controllable,} \\ & \text{and } \sum_{l=1}^{p}|\mathcal{J}_l| \leq \sum_{l=1}^{p}|\mathcal{J}'_l|, \text{ for any} \\ & \text{struct. controllable } (\bar{A}, \mathbb{I}_n(\mathcal{J}'_1, \ldots, \mathcal{J}'_p)).\end{aligned} \qquad (5)$$

◇

2. PRELIMINARIES AND TERMINOLOGY

In this section, we review some basic concepts of structural systems and graph theory, followed by concepts of computational complexity. In addition, we introduce terminology that will be employed throughout the rest of the paper.

Consider a linear time-invariant (LTI) system described by the pair (A, B). In order to perform structural analysis efficiently, it is customary to associate to (4) a directed graph (digraph) $\mathcal{D} = (\mathcal{V}, \mathcal{E})$, in which $\mathcal{V}$ denotes the set of *vertices* and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ the set of *edges*, where (v_j, v_i) represents an edge from the vertex v_j to vertex v_i. To this end, let $\bar{A} \in \{0,1\}^{n\times n}$ and $\bar{B} \in \{0,1\}^{n\times p}$ be binary matrices that represent the sparsity patterns of A and B, respectively. Denote by $\mathcal{X} = \{x_1, \ldots, x_n\}$ and $\mathcal{U} = \{u_1, \ldots, u_p\}$ the sets of state and input vertices, respectively. And by $\mathcal{E}_{\mathcal{X},\mathcal{X}} = \{(x_i, x_j) : \bar{A}_{ji} \neq 0\}$ and $\mathcal{E}_{\mathcal{U},\mathcal{X}} = \{(u_j, x_i) : \bar{B}_{ij} \neq 0\}$ the edges between the sets in subscript. In addition, we introduce the *state digraph* $\mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$, and the *system digraph* $\mathcal{D}(\bar{A}, \bar{B}) = (\mathcal{X} \cup \mathcal{U}, \mathcal{E}_{\mathcal{X},\mathcal{X}} \cup \mathcal{E}_{\mathcal{U},\mathcal{X}})$. Further, by similarity, we have the *state-slack digraph* given by $\mathcal{D}(\bar{A}, \bar{S}) = (\mathcal{X} \cup \mathcal{S}, \mathcal{E}_{\mathcal{X},\mathcal{X}} \cup \mathcal{E}_{\mathcal{S},\mathcal{X}})$, where $\mathcal{S}$ represents the set of slack variables (or vertices), which may be thought of as potential inputs. In addition, given digraphs $\mathcal{D}(\bar{A}, \bar{B})$ and $\mathcal{D}(\bar{A}, \bar{S})$, we say that they are *isomorphic* to each other, if there exists a bijective relationship between the vertices and edges of the digraphs that preserves the incidence relation. Finally, since the edges are directed, an edge is said to be an *outgoing edge* from a vertex v if it starts in v, and, similarly, is said to be an *incoming edge* to w if it ends on w.

A *directed path* between the vertices v_1 and v_k is a sequence of edges $\{(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k)\}$. If all the vertices in a directed path are different, then the path is said to be an *elementary path*.

We also require the following graph-theoretic notions (Cormen et al. (2001)): A digraph is strongly connected if there exists a directed path between any two vertices. A *strongly connected component* (SCC) is a maximal subgraph $\mathcal{D}_S = (\mathcal{V}_S, \mathcal{E}_S)$ of $\mathcal{D}$, i.e., a graph comprising a set of vertices $\mathcal{V}' \subset \mathcal{V}$ and of edges $\mathcal{E}' \subset \mathcal{E}$, such that for every $u, v \in \mathcal{V}_S$ there exists a path from u to v and is maximal with this property (i.e., considering any other vertex will make the subgraph cease to be strongly connected).

Since the SCCs of a digraph $\mathcal{D} = (\mathcal{V}, \mathcal{E})$ are uniquely determined, we can regard each SCC as a virtual node. By doing so we build a *directed acyclic graph* (DAG), i.e., a directed graph with no cycles, in which a directed edge exists between two virtual nodes representing two SCCs *if and only if* there exists an edge between two vertices in the corresponding SCCs in the original digraph. We call this the DAG representation of the graph, which can be computed efficiently in $\mathcal{O}(|\mathcal{V}| + |\mathcal{E}|)$ (Cormen et al. (2001)). We can further classify the SCCs with respect to the existence of incoming and/or outgoing edges as follows.

Definition 1. (Pequito et al. (2014b)). An SCC is said to be linked if it has at least one incoming or outgoing edge from another SCC. In particular, an SCC is *non top-linked* if it has no incoming edges from another SCC. ⋄

For any digraph $\mathcal{D} = (\mathcal{V}, \mathcal{E})$ and any two vertex sets $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{V}$ we define the *bipartite graph* $\mathcal{B}(\mathcal{S}_1, \mathcal{S}_2, \mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2})$ whose vertex set is given by $\mathcal{S}_1 \cup \mathcal{S}_2$ and the edge set $\mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2} = \mathcal{E} \cap (\mathcal{S}_1 \times \mathcal{S}_2)$. We call the bipartite graph $\mathcal{B}(\mathcal{V}, \mathcal{V}, \mathcal{E})$ the bipartite graph associated with $\mathcal{D}(\mathcal{V}, \mathcal{E})$. In the sequel we will make heavy use of the *state bipartite graph* $\mathcal{B}(\bar{A}) \equiv \mathcal{B}(\mathcal{X}, \mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$, which is the bipartite graph associated with the state digraph $\mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$. Similarly, we have the *state-slack bipartite graph* $\mathcal{B}(\bar{A}, \bar{S}) = \mathcal{B}(\mathcal{X} \cup \mathcal{S}, \mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}} \cup \mathcal{E}_{\mathcal{S},\mathcal{X}})$ that we refer to as the bipartite graph associated with the state-slack digraph $\mathcal{D}(\bar{A}, \bar{S})$.

Given $\mathcal{B}(\mathcal{S}_1, \mathcal{S}_2, \mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2})$, a matching M corresponds to a subset of edges in $\mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2}$ so that no two edges have a vertex in common, (i.e., given edges $e = (s_1, s_2)$ and $e' = (s_1', s_2')$ with $s_1, s_1' \in \mathcal{S}_1$ and $s_2, s_2' \in \mathcal{S}_2$, $e, e' \in M$ only if $s_1 \neq s_1'$ and $s_2 \neq s_2'$). We say that a matching M^* is a *maximum matching* of $\mathcal{B}$ if no other matching of $\mathcal{B}$ contains more edges that M^*. Note that in general maximum matchings are not unique.

We call the vertices in $\mathcal{S}_1$ and $\mathcal{S}_2$ belonging to an edge in a matching M the *matched vertices* with respect to (w.r.t.) M and *unmatched vertices* otherwise. For ease of referencing, the term *right-unmatched vertices* associated with the matching M of $\mathcal{B}(\mathcal{S}_1, \mathcal{S}_2, \mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2})$ (not necessarily maximum) will refer to those vertices in $\mathcal{S}_2$ that do not belong to a matching edge in M.

If we associate *weights* (or costs) with the edges in a digraph and bipartite graph, we obtain a *weighted digraph* and *weighted bipartite graph*, respectively. A weighted digraph is represented by the digraph-weight pair given by $(\mathcal{D} = (\mathcal{V}, \mathcal{E}); w)$, where $w : \mathcal{E} \to \mathbb{R}_0^+ \cup \{\infty\}$ is the weight function. Similarly, a weighted bipartite graph is represented by the bipartite-weight pair $(\mathcal{B}(\mathcal{S}_1, \mathcal{S}_2, \mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2}); w)$.

Subsequently, we introduce the *minimum weight maximum matching* problem. This problem consists in determining the maximum matching of a weighted bipartite graph $(\mathcal{B}(\mathcal{S}_1, \mathcal{S}_2, \mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2}); w)$ that incurs the minimum weight-sum of its edges; in other words, determining the maximum matching M^c such that

$$M^c = \arg\min_{M\in\mathcal{M}} \sum_{e\in M} w(e),$$

where $\mathcal{M}$ is the set of all maximum matchings of $\mathcal{B}(\mathcal{S}_1, \mathcal{S}_2, \mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2})$. This problem can be efficiently solved using the Hungarian algorithm (Munkres (1957)) with computational complexity of $\mathcal{O}(\max\{|\mathcal{S}_1|, |\mathcal{S}_2|\}^3)$.

We will also require the following general results on structural control design from (Pequito et al. (2014b); Pequito et al. (2013c)). We define a *feasible dedicated input configuration* to be a collection of state variables to which by assigning dedicated inputs we can ensure structural controllability of the system. Consequently, a minimal feasible dedicated input configuration is the minimal subset of state variables to which we need to assign dedicated inputs to ensure structural controllability. Further, the feasible dedicated input configurations can be characterized as follows.

Theorem 1. (Pequito et al. (2014b)). Denote the system digraph $\mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ and the associated state bipartite graph $\mathcal{B}(\bar{A}) \equiv \mathcal{B}(\mathcal{X}, \mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$. Let $\mathcal{S}_u \subset \mathcal{X}$, then the following statements are equivalent:

(1) The set $\mathcal{S}_u$ is a feasible dedicated input configuration;
(2) There exists a subset $\mathcal{U}_R(M^*) \subset \mathcal{S}_u$ corresponding to the set of right-unmatched vertices of some maximum matching M^* of $\mathcal{B}(\bar{A})$, and a subset $\mathcal{A}_u \subset \mathcal{S}_u$ comprising one state variable from each non top-linked SCC of $\mathcal{D}(\bar{A})$. ⋄

Finally, we will require the following general results regarding maximum matching properties.

Lemma 1. (Pequito et al. (2014a)). Let $\mathcal{B}(\bar{A},\bar{S}) = \mathcal{B}(\mathcal{X} \cup \mathcal{S}, \mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}} \cup \mathcal{E}_{\mathcal{S},\mathcal{X}})$ be the state-slack bipartite graph. If $M^*_{\bar{A},\bar{S}}$ is a maximum matching of $\mathcal{B}(\bar{A},\bar{S}) = \mathcal{B}(\mathcal{X} \cup \mathcal{S}, \mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}} \cup \mathcal{E}_{\mathcal{S},\mathcal{X}})$, then $M^*_{\bar{A},\bar{S}} = M_{\bar{S}} \cup M_{\bar{A}}$, where $M_{\bar{A}} = M_{\bar{A},\bar{S}} \cap \mathcal{E}_{\mathcal{X},\mathcal{X}}$ and $M_{\bar{S}} = M_{\bar{A},\bar{S}} \cap \mathcal{E}_{\mathcal{S},\mathcal{X}}$ are (disjoint) matchings of $\mathcal{B}(\bar{A})$ and $\mathcal{B}(\bar{S})$, respectively, and $M_{\bar{S}}$ contains the largest collection of edges incoming into a set of right-unmatched vertices of some maximum matching of $\mathcal{B}(\bar{A})$. In particular, $\mathcal{R}(M_{\bar{S}}) \subset \mathcal{U}_R(M_{\bar{A}})$, where $\mathcal{U}_R(M_{\bar{A}})$ is the set of right-unmatched vertices associated with the (possibly not maximum) matching $M_{\bar{A}}$. ⋄

Lemma 2. (Pequito et al. (2014a)). Let $\bar{A} \in \{0,1\}^{n\times n}$ and $\bar{S} \in \{0,1\}^{n\times p}$ with $p \leq n$. Consider the weighted state-slack bipartite graph $(\mathcal{B}(\bar{A},\bar{S}); w)$, where $\mathcal{B}(\bar{A},\bar{S}) = \mathcal{B}\big(\mathcal{X} \cup \mathcal{S}, \mathcal{X}, \mathcal{E} \equiv (\mathcal{E}_{\mathcal{X},\mathcal{X}} \cup \mathcal{E}_{\mathcal{S},\mathcal{X}})\big)$, and $w : \mathcal{E} \to \mathbb{R}^+_0 \cup \{\infty\}$ such that $w(e_{\bar{S}}) < w(e_{\bar{A}}) = c_{\bar{A}} \in \mathbb{R}^+$, with $e_{\bar{S}} \in \mathcal{E}_{\mathcal{S},\mathcal{X}}$ and $e_{\bar{A}} \in \mathcal{E}_{\mathcal{X},\mathcal{X}}$. A minimum weighted maximum matching $M^*_{\bar{A},\bar{S}}$ of $(\mathcal{B}(\bar{A},\bar{S}); w)$ is given by

$$M^*_{\bar{A},\bar{S}} = M^*_{\bar{S}} \cup M_{\bar{A}},$$

where $M^*_{\bar{S}}$ and $M_{\bar{A}}$ are as given in Lemma 1, and $M^*_{\bar{S}}$ is a maximum matching of $\mathcal{B}(\bar{S}) = \mathcal{B}(\mathcal{S},\mathcal{X},\mathcal{E}_{\mathcal{S},\mathcal{X}})$ whose edges incur the lowest weight-sum among all possible maximum matchings of $\mathcal{B}(\bar{S})$. ⋄

3. MAIN RESULTS

In this section, we present the main result of this paper. More precisely, we present the reduction of $\mathcal{P}_1$ to a minimum weight maximum matching problem. Intuitively, given the system's dynamical structure and its digraph representation, we consider an extended digraph with as many slack variables as the minimum number of state variables required to obtain a feasible dedicated input configuration. These slack variables will indicate which state variables should be actuated to achieve a such an input configuration. Towards this goal, outgoing edges from the slack variables into the state variables (to be considered for the feasible dedicated input configuration) are judiciously chosen such that a minimum weight maximum matching containing these edges exists, hence corresponding to the feasible dedicated input configuration that incurs the minimum cost. The systematic reduction of $\mathcal{P}_1$ to a minimum weight maximum matching problem is presented in Algorithm 1. Next, we present the proof of correctness of Algorithm 1 and its complexity.

Theorem 2. Algorithm 1 is correct, i.e., it provides a solution to $\mathcal{P}_1$ (as long as the set of feasible $\mathbb{I}(\mathcal{J}_1,\ldots,\mathcal{J}_p)$'s is non-empty). Moreover, its computational complexity is $\mathcal{O}(n^3)$. ⋄

Proof. The proof follows by first showing feasibility of the solution obtained using Algorithm 1, and, secondly, showing that it is minimal, which will be proved by contradiction. To see that the solution obtained using Algorithm 1 is feasible, we need to verify that $\mathbb{I}_n(\mathcal{J}_1,\ldots,\mathcal{J}_p)$ is such that $(\bar{A}, \mathbb{I}(\mathcal{J}_1,\ldots,\mathcal{J}_p))$ is structurally controllable, if $|\mathcal{J}_1|+\ldots+|\mathcal{J}_p| = \alpha$ and the weight-sum of $M^*_{\bar{A},\bar{B}}$ is finite. Towards this goal, let $\mathcal{D}(\bar{A})$ be the state digraph comprising β non top-linked SCCs, and a minimal feasible dedicated input configuration be of size $\alpha \geq \beta$, determined using a $\mathcal{O}(n^3)$ algorithm (Pequito et al. (2014b)). Now, consider an augmented digraph $\mathcal{D}(\bar{A},\bar{S})$, with $\bar{S} \in \{0,1\}^{n\times\alpha}$, i.e., with α slack variables, that will indicate the variables to be considered for obtaining a minimal feasible dedicated input configuration. In addition, $\bar{S}$ satisfies the following conditions: each of the β slack variables are such that slack variable k, with $k = 1,\ldots,\beta$, has only outgoing edges to all the state variables in the non top-linked SCC k, and the remaining $\alpha-\beta$ slack variables have edges to all state variables. From Lemma 1 and the knowledge that a feasible dedicated input configuration with α state variables exists, we can argue that a maximum matching of $\mathcal{B}(\bar{A},\bar{S})$ contains edges outgoing from slack variables and ending in all right-unmatched vertices with respect to a maximum

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Algorithm 1 Solution to $\mathcal{P}_1$

Input: The structural $n \times n$ system matrix $\bar{A}$, and the vector c^i of size n comprising the cost of actuating each state variable by actuator i.

Output: A solution $\mathbb{I}_n(\mathcal{J}_1^*,\ldots,\mathcal{J}_p^*)$ to $\mathcal{P}_1$.

1. Determine the minimum number α of dedicated inputs required to ensure structural controllability (Pequito et al. (2014b)).

2. Let $\mathcal{N}_j^T$, with $j = 1,\cdots,\beta$, denote the non top-linked SCCs of $\mathcal{D}(\bar{A})$. Let $c_{\max}$ be the maximum real value (i.e., not considering ∞) in c^i for all $i \in \{1,\ldots,p\}$, and consider α slack variables, where each slack variable $k = 1,\ldots,\beta$ has outgoing edges to all the state variables in the k-th non top-linked SCC $\mathcal{N}_k^T$, whereas, for the remaining $\alpha-\beta$ slack variables have outgoing edges to all state variables, i.e.,

$$\bar{S} = \begin{bmatrix} | & | & & | \\ \bar{s}_1 & \bar{s}_2 & \cdots & \bar{s}_\alpha \\ | & | & & | \end{bmatrix}.$$

For $k = 1,\ldots,\beta$, the ith entry of $\bar{s}_k$ is given by $[\bar{s}_k]_i = 1$ if $x_i \in \mathcal{N}^k$ and $[\bar{s}_k]_i = 0$ otherwise. For $k = \beta+1,\ldots,p$ we have $[\bar{s}_k]_i = 1$ for $i = 1,\ldots,n$. Now, consider $(\mathcal{B}(\bar{A},\bar{S}); w)$ where w is given as follows:

$$w(e) = \begin{cases} c_{\max}+1, & e \in \mathcal{E}_{\mathcal{X},\mathcal{X}}, \\ \min\limits_{k\in\{1,\ldots,p\}} c_i^k, & e \equiv (s_j, x_i) \in \mathcal{E}_{\mathcal{S},\mathcal{X}},\ j = 1,\ldots,\alpha, \\ \infty, & \text{otherwise.} \end{cases}$$

3. Determine the minimum weight maximum matching M^* associated with the bipartite graph $(\mathcal{B}(\bar{A},\bar{S}); w)$.

4. Let $\mathcal{I}_i = \{l : l \equiv \arg\min_{k\in\{1,\ldots,p\}} c_i^k, (s,x_i) \in M^*\}$, with $i = 1,\ldots,n$, be the index of the actuator incurring the smallest cost actuating state variable x_i. In addition, we assume that l consists of one element of $\arg\min_{k\in\{1,\ldots,p\}} c_i^k$ if the set is non-empty, since in general there can exist different minimum c_i^k, which implies that the set $\arg\min_{k\in\{1,\ldots,p\}} c_i^k$ contains more than one element.

5. Let $\mathcal{J}_j = \{i : \mathcal{I}_i \text{ contains } j\}$, i.e., the collection of dedicated inputs in actuator j that effectively actuate the state variable x_i. If $|\mathcal{J}_1| + \ldots + |\mathcal{J}_p| = \alpha$ and the weight-sum of M^* is finite, then $(\bar{A}, \mathbb{I}(\mathcal{J}_1,\ldots,\mathcal{J}_p))$ is structurally controllable, and a solution to $\mathcal{P}_1$ is obtained; otherwise, the problem is infeasible, i.e., there is no feasible $\mathbb{I}(\mathcal{J}_1,\ldots,\mathcal{J}_p)$ (with finite cost) such that $(\bar{A}, \mathbb{I}(\mathcal{J}_1,\ldots,\mathcal{J}_p))$ is structurally controllable.

matching of $\mathcal{B}(\bar{A})$. Additionally, there exists a maximum matching $M^*_{\bar{A},\bar{S}}$ of $\mathcal{B}(\bar{A},\bar{S})$, where all slack variables belong to matching edges in $M^*_{\bar{A},\bar{S}}$. In the former case, due to the proposed construction, there is at least one edge from a slack variable to each non top-linked SCC. Hence, by Theorem 1, the collection of the state variables, where the edges with origin in slack variables belonging to $M^*_{\bar{A},\bar{S}}$ end, is a feasible dedicated input configuration; such a collection is also minimal since it has exactly α state variables – the size of a minimal feasible dedicated input configuration. Therefore, we aim to determine such a matching, which will be accomplished by considering a minimum weight maximum matching problem with the proposed weight function w. Therefore, taking $(\mathcal{B}(\bar{A},\bar{S});w)$ to be the weighted version of $\mathcal{B}(\bar{A},\bar{S})$, by invoking Lemma 2, there exists a maximum matching $M^*_{\bar{A},\bar{S}}$ of $\mathcal{B}(\bar{A},\bar{S})$, where each edge with origin in slack variables belonging to $M^*_{\bar{A},\bar{S}}$ indicates which state variables should be actuated, and such a collection is a feasible dedicated input configuration if it is of size p and the sum of the weights of the edges in $M^*_{\bar{A},\bar{S}}$ is finite. Alternatively, an infinite cost would correspond to the case where no feasible dedicated input configuration exists, i.e., no finite cost input matrix $\mathbb{I}(\mathcal{J}_1,\ldots,\mathcal{J}_p)$ can make the system structurally controllable. In summary, we obtain a minimal feasible dedicated input configuration with the lowest cost, which corresponds to a (dedicated) solution to $\mathcal{P}_1$.

To show that $\mathbb{I}_n(\mathcal{J}_1,\ldots,\mathcal{J}_p)$ obtained by Algorithm 1 incurs the minimum cost, suppose by contradiction that this is not the case. So, assume that there exists another feasible $\mathbb{I}_n(\mathcal{J}'_1,\ldots,\mathcal{J}'_p)$ leading to lower cost. Therefore, by letting $\mathcal{D}(\bar{A},\mathbb{I}_n(\mathcal{J}'_1,\ldots,\mathcal{J}'_p)) = (\mathcal{X}\cup\mathcal{U},\mathcal{E}_{\mathcal{X},\mathcal{X}}\cup\mathcal{E}_{\mathcal{U},\mathcal{X}})$ and $\mathcal{D}(\bar{A},\bar{S})$ to be isomorphic, and considering the weight function w as in Algorithm 1, it follows by Lemma 2 that there exists a maximum matching M' of $(\mathcal{B}(\bar{A},\bar{S}) = (\mathcal{X}\cup\mathcal{S},\mathcal{E}_{\mathcal{X},\mathcal{X}}\cup\mathcal{E}_{\mathcal{S},\mathcal{X}});w)$ containing $\mathcal{E}_{\mathcal{S},\mathcal{X}}$. Nevertheless, this is a contradiction since it implies that there exists a maximum matching M' incurring a lower cost than M^* obtained using, for instance, the Hungarian algorithm (Munkres (1957)), and used to construct $\mathbb{I}_n(\mathcal{J}_1,\ldots,\mathcal{J}_p)$.

The computational complexity follows from noticing that Step 1 has complexity $\mathcal{O}(n^3)$ (Pequito et al. (2014b)). Step 2 can be computed using linear complexity algorithms. In Step 3, the Hungarian algorithm is used on the $n\times(n+p)$ matrix obtained at the end of Step 2, and incurs $\mathcal{O}(n^3)$ complexity. Finally, Step 4 consists of a for-loop operation which has linear complexity. Hence, summing up the different complexities, the result follows. ∎

4. ILLUSTRATIVE EXAMPLES

In this section, we provide two examples illustrating different scenarios where the main results derived in Section 3 can be used.

4.1 Example 1

Consider the scenario depicted in Figure 1, where three towers communicate with all four agents. Let the communication cost be described as follows: $c^1 = [\,1\ 3\ 2\ 3\,]$, $c^2 = [\,2\ 1\ 2\ 3\,]$ and $c^3 = [\,3\ 3\ 2\ 1\,]$. Recall Algorithm 1, and notice that the minimal feasible dedicated input configuration consists of two state variables (see Step 1), and the state digraph depicted in Figure 1 b) consists in a single SCC. Thus, $\bar{S}$ is the 4×2 matrix with all entries equal to one. In addition, the weights $w(e)$ are defined as in Step 2, where $c_{\max} = 3$. There are three possible maximum matchings in Step 3: $M^1 = \{(s_1,x_1),(s_2,x_2),(x_3,x_4),(x_4,x_3)\}$, $M^2 = \{(s_3,x_3),(s_2,x_2),(x_3,x_1),(x_1,x_3)\}$ and $M^3 = \{(s_1,x_1),(s_3,x_3),(x_1,x_2),(x_2,x_1)\}$. For illustrative purposes, take $M^* \equiv M^1$. Therefore, in Step 4, we obtain $\mathcal{I}_1 = \{1\}$, $\mathcal{I}_2 = \{2\}$, $\mathcal{I}_3 = \emptyset$ and $\mathcal{I}_4 = \emptyset$. Thus, $\mathcal{J}_1 = \{1\}$, $\mathcal{J}_2 = \{2\}$ and $\mathcal{J}_3 = \emptyset$. Subsequently, by performing Step 5, it is easy to see that $|\mathcal{J}_1|+|\mathcal{J}_2|+|\mathcal{J}_3| = 2$ and the weight-sum of M^* equals 10, since it corresponds to the sum of the following weights: $w((s_1,x_1)) = w((s_2,x_2)) = 2$ and $w((x_3,x_4)) = w((x_4,x_3)) = 4$. Finally, we notice that the final solution has dedicated inputs from tower 1 to agent 1, and from tower 2 to agent 2.

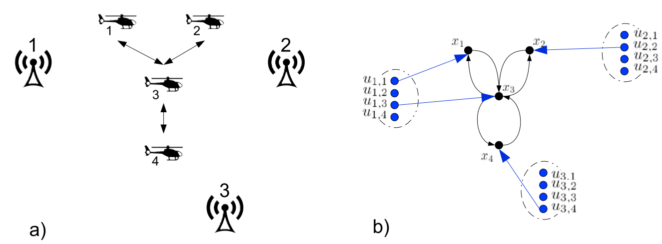

Fig. 1. In a) we show a configuration of four agents with bidirectional communication and control towers that can send inputs to all agents. The communication between a tower and an agent incurs a cost that is proportional to the square of the distance. Therefore, in b) we depict in black the state digraph associated with the dynamics induced by the communication topology of the agents, which consists of a single SCC, and in blue the actuation capabilities of the different towers. We note that there should be depicted an outgoing edge from each blue vertex $u_{i,j}$ to the state vertex x_j, which will make the figure difficult to read. Therefore we illustrate with the edges corresponding to actuation from the closest tower.

4.2 Example 2

Consider the scenario depicted in Figure 2, where each of the three towers communicates with only a subset of the three agents. Let the communication cost be described as follows: $c^1 = [\,\infty\ 1\ 1\,]$, $c^2 = [\,\infty\ \infty\ 1\,]$ and $c^3 = [\,1\ 1\ \infty\,]$. We note that there only exists one non top-linked SCC, namely that consisting of vertices x_1 and x_2. In a maximum matching of the state bipartite graph, we see that there is one right-unmatched vertex, which, depending on the matching, may be either vertex x_1 or x_3. Since x_1 is a right-unmatched vertex for a maximum matching and the only non top-linked SCC, only one control input is required to ensure structural controllability, see Theorem 1. Therefore, we introduce one slack variable and connect it to vertices x_1 and x_2 as prescribed in Step 2 in Algorithm 1. The minimum costs of actuating these two state vertices, and thus the weights of the edges from slack vertices, are both 1. Note, however, that state vertex x_1 can only be actuated by control tower 3, while vertex x_2 may be actuated by control towers 1 or 3. Since the greatest cost in the c^i's (not including ∞) is 1, we have $c_{\max} = 2$. The maximum matching of the state-slack bipartite graph includes edges (s_1, x_1), (x_1, x_2), and (x_2, x_3). Thus, the optimal solution is that tower 3 will actuate state vertex x_1.

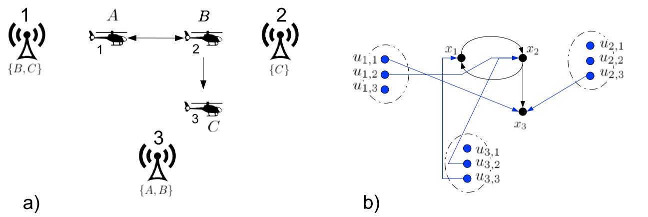

Fig. 2. In a) we show a configuration of agents $\{A, B, C\}$ and communication towers, each of which can only provide information to a subset (labeled at its base) of the agents. In b) we depict in black the state digraph associated with the dynamics induced by the communication topology of the agents. The state digraph is composed of two SCCs, $\{x_1, x_2\}$ and $\{x_3\}$, where the former is a non top-linked SCC. In blue we depict the actuation capability of the different communication towers, where the blue vertices are all actuation capabilities, but only those with outgoing blue edges correspond to the feasible actuation – imposed by the communication constraints from the towers to the agents.

5. CONCLUSIONS AND FURTHER RESEARCH

In this paper, we provided a solution to the sparsest minimum multiple-cost structural leader selection problem: the problem of determining the minimum number of leaders selected to control a network of agents when input signal can originate in different sources, for instance, communication/control towers, and incur different costs. Possible extensions include, but are not limited to, considering scenarios where the communication between agents is time-varying and the costs are also time-varying due to updates in agent locations. Additionally, it would be of interest to explore how the current results can be extended to agents with arbitrary state space dimension. Finally, it would be interesting to explore practical applications, for instance, swarm formation.

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