

Control-aware Random Access Communication

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Abstract—In modern control applications multiple wireless sensors need to efficiently share the available wireless medium to communicate with their respective actuators. Random access policies, where each sensor independently decides whether to access the shared wireless medium, are attractive as they do not require central coordination. However interference between simultaneous transmissions causes transmitted packets to collide, leading to control performance degradation or potentially instability. Given plant and controller dynamics, we derive a sufficient condition for the access policy employed by each sensor so that wireless interference does not violate stability of any involved control loop. Based on this decoupling condition we design random access communication policies that are control-aware by adapting to the physical plant states measured by the sensors online. The control performance of our design is illustrated in numerical simulations.

I. INTRODUCTION

The abundance of wireless sensing devices in modern control environments, for example, smart homes or modern urban infrastructures, creates a need for efficiently sharing the available wireless medium. The prevalent approach to the problem of sharing a wireless medium in networked control systems has been centralized scheduling. In such a setup the wireless sensors/actuators either communicate in a predesigned periodic sequence [1]–[3], or a central authority schedules dynamically which device accesses the channel at each time step [4]–[6]. On the contrary, decentralized mechanisms allow for easier practical implementations but the uncoordinated channel access may introduce control performance degradation or loss of stability.

We examine a setup where multiple control loops share a wireless medium using random access communication [7, Ch.14], which is a simple decentralized mechanism. Each sensor independently decides whether to access the channel and transmit to its controller to close the loop. This mechanism is easy to implement as it does not require predesigned sequences of how sensors access the medium or a central scheduling authority. The challenge however is that interference arises in the wireless channel if more than one sensors transmit simultaneously, leading to messages being lost (packet collision) and systems running in open loop.

The problem of designing random access communication or other decentralized contention-based mechanisms suitable for control systems has drawn limited attention, to the best of our knowledge. The focus had been primarily on analyzing

the impact of packet collisions for networked control systems and on comparing different medium access mechanisms, either numerically [8]–[10] or analytically in simple cases [11]–[13]. General conditions for stability under packet collisions have also been examined in [14]. Besides closed loop control, optimal remote estimation over collision channels is considered recently in [15].

In contrast in this paper we examine how to systematically design random access mechanisms suitable for closed loop control. We assume a controller for each system is given and we consider general control-aware random access policies for the sensors. Each sensor may use available information about its current or past plant state measurements to decide whether to transmit (Section II). This information may be useful for example to avoid transmitting as long as its corresponding plant state is close to the desirable operating point, in an effort to limit packet collisions to other control loops. Preliminary designs of random access policies for control systems were considered in our previous work [16], [17], where the policies were simpler and non-adapted to sensor measurements in contrast to the online policies of the present work. It was shown that each sensor should access the channel at a rate proportional to the desired control performance of its loop, and inverse proportional to the aggregate collision effect it has on all other control loops.

State-based communication policies in the absence of wireless interference have been popular within the event-based control framework [12], [18]–[22] which explores the idea of not controlling/actuating on a plant as long as the plant state does not exceed some threshold (i.e., an event occurs). Event-based control is shown to provide desirable closed-loop performance with communication rates lower than in standard periodic control setups. Here in contrast our goal is to exploit plant-state information so that no disastrous interference is caused on other control loops over the shared channel. The emerging collisions introduce coupling between the evolutions of the plants, making analysis hard. This coupling has been documented in other works considering state-dependent mechanisms for contention-based channel access [9], [10], [12], [13]. The analysis however is simplified in these works by focusing mainly on simple dynamics (e.g., scalar, identical systems) or by approximating the packet collisions imposed on one control loop from all others as i.i.d packet drops [12]. Simulation-based analysis and design is considered in [10].

In this paper we take an alternative approach to address the coupling between control loops over the shared wireless channel and to design control-aware random access policies. Our main contribution is to derive a sufficient mathematical condition for the access policy of each sensor so that it does

This work was supported in part by NSF CNS-0931239, and by TerraSwarm, one of six centers of STARnet, a Semiconductor Research Corporation program sponsored by MARCO and DARPA.

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not violate the stabilizability of any other loop (Section III). Decoupling between all access policies is achieved since the condition is expressed for each sensor separately. Verifying the condition is computationally challenging and we discuss some approaches for addressing this issue.

We then proceed in Section IV to design each sensor's control-aware policy so that the above condition holds. The policies we propose adapt to the sensor measurements online and are threshold-like, similar to standard event-based policies [18]–[20]. The distinguishing feature of our policies is that the sensors change their thresholds dynamically over time in a way that by design mitigates the interference between loops over the shared medium. The control performance of the proposed policies is evaluated in numerical simulations in Section V. We conclude with some remarks in Section VI.

Notation: We denote the space of $n \times n$ symmetric positive definite matrices by \mathbb{S}_{++}^n . We also denote by \succ the comparison with respect to the positive definite cone.

II. PROBLEM DESCRIPTION

We consider a wireless control architecture where m independent plants are controlled over a shared wireless medium. Each sensor i ($i = 1, 2, \dots, m$) measures the state of plant i and transmits it to a corresponding controller i computing the plant control input. Packet collisions might arise on the shared medium between simultaneously transmitting sensors. The case for $m = 2$ control loops is shown in Fig. 1. We are interested in designing a decentralized mechanism for each sensor to decide whether to access the medium (random access) in a way that stabilizability can be guaranteed for all control systems.

Communication takes place in time slots. At every time k each sensor i randomly and independently decides to access the channel with some probability $\alpha_{i,k} \in [0, 1]$, which is our design variable. If only sensor i transmits at a time slot, the message is received at its corresponding controller and the i th loop successfully closes at this time step. To model the interference in the shared wireless medium, we suppose that if more than one sensors transmit at the same slot, a collision occurs on all sent packets. This model is often employed in control literature [8], [14] and in wireless communication networks [23], [24] – see Remark 1 for extensions to more general channel models. Let us indicate with $\gamma_{i,k} \in \{0, 1\}$ the success of the transmission at time slot k for link/system i . This is a Bernoulli random variable with success probability

$$\mathbb{P}(\gamma_{i,k} = 1) = \alpha_{i,k} \prod_{j \neq i} (1 - \alpha_{j,k}). \quad (1)$$

This expression states that the probability of system i closing the loop at time k equals the probability that sensor i transmits, multiplied by the probability that no other sensor $j \neq i$ is causing collisions on i th transmission.

Our goal is to design the communication aspects of the problem, hence we assume the dynamics for all m control

¹Unless otherwise noted we use $j \neq i$ throughout this paper to refer to sensors j belonging in the set $\{1, \dots, m\} \setminus \{i\}$.

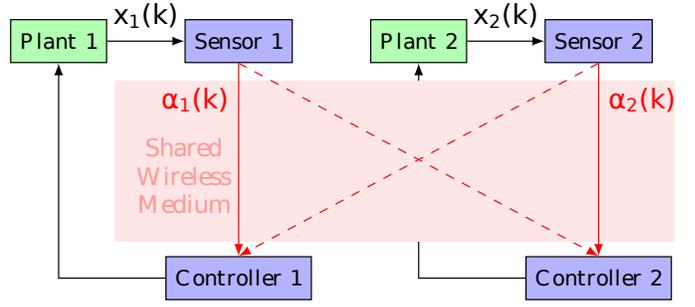


Fig. 1. Random access architecture over a shared wireless medium for $m = 2$ control loops. Each i sensor measures its respective plant state $x_{i,k}$ and randomly accesses the shared wireless medium with probability $\alpha_{i,k}$ to transmit to a corresponding controller computing the plant inputs. Packet collisions occur when both sensors transmit at the same slot. The goal of a decentralized channel access mechanism is to guarantee stability for all control loops.

systems are fixed, meaning that controllers have been already designed. We suppose the system evolution is described by a switched linear time invariant model of the form

$$x_{i,k+1} = \begin{cases} A_{c,i} x_{i,k} + w_{i,k}, & \text{if } \gamma_{i,k} = 1 \\ A_{o,i} x_{i,k} + w_{i,k}, & \text{if } \gamma_{i,k} = 0 \end{cases} \quad (2)$$

Here $x_{i,k} \in \mathbb{R}^{n_i}$ denotes the state of control system i at each time k , which can in general include both plant and controller states – see, e.g., [25]. At a successful transmission the system dynamics are described by the matrix $A_{c,i} \in \mathbb{R}^{n_i \times n_i}$, where ‘c’ stands for closed-loop, and otherwise by $A_{o,i} \in \mathbb{R}^{n_i \times n_i}$, where ‘o’ stands for open-loop. We assume that $A_{c,i}$ is asymptotically stable, implying that if system i successfully transmits at each slot the state evolution of $x_{i,k}$ is stable. The open loop matrix $A_{o,i}$ may be unstable. The additive terms $w_{i,k}$ model an independent (both across time k for each system i , and across systems) identically distributed (i.i.d.) Gaussian noise process with mean zero and positive definite covariance W_i . We emphasize that knowledge of the distribution is important in our approach – see Sec. IV later and Remark 2 for a discussion on non-Gaussian disturbances. An example of a networked control system of the form (2) is presented next.

Example 1. Suppose each closed loop i consists of a linear plant of the form

$$x_{i,k+1} = A_i x_{i,k} + B_i u_{i,k} + w_{i,k}, \quad (3)$$

where $w_{i,k}$ is an i.i.d. Gaussian disturbance. Each wireless sensor i measures the state $x_{i,k}$ and decides whether to transmit it to the controller. Consider a simple control law which applies a zero input $u_{i,k} = 0$ when no information is received, and upon receiving a measurement it applies a state feedback $u_{i,k} = K_i x_{i,k}$ leading to a stable closed loop mode $A_i + B_i K_i$. The overall networked system dynamics are expressed as

$$x_{i,k+1} = \begin{cases} (A_i + B_i K_i) x_{i,k} + w_{i,k}, & \text{if } \gamma_{i,k} = 1 \\ A_i x_{i,k} + w_{i,k}, & \text{if } \gamma_{i,k} = 0 \end{cases} \quad (4)$$

which is of the form (2). \square

The interference due to the uncoordinated transmissions over the shared wireless medium couples the evolution of all the systems. The access rate $\alpha_{i,k} \in [0, 1]$ selected by each sensor $i = 1, \dots, m$ affects the success of transmissions over all links by (1), and consequently the switch between the two modes of operation (open and closed loop) in (2) for all systems at each time step.

Our goal is to design how each sensor selects its channel access probability adapting online to its own plant state measurements. In general sensor i may choose $\alpha_{i,k}$ at time k based on the current measurement $x_{i,k}$ and any information collected so far, i.e., all previous local measurements $x_{i,0}, \dots, x_{i,k-1}$ as well as the success of all previous transmissions $\gamma_{i,0}, \dots, \gamma_{i,k-1}$. The latter can be made available by acknowledgments sent from the receiver/controller and we assume perfect acknowledgments without collisions. To keep the implementation simple we assume each sensor i does not have any information about other loops $j \neq i$.

Before measuring the state $x_{i,k}$, the sensor i knows by (2) that the measurement has a Gaussian distribution with some mean $\mu_{i,k}$ and variance W_i . The mean $\mu_{i,k}$ depends on whether a successful transmission occurred at the last step. If it did, the sensor knows that the plant state evolved according to the closed loop mode $A_{c,i}$ and expects a measurement $x_{i,k}$ with mean $\mu_{i,k} = A_{c,i}x_{i,k-1}$. The case of no transmission can be similarly argued. The mean $\mu_{i,k}$ then is defined as

$$\mu_{i,k} = \begin{cases} A_{c,i}x_{i,k-1}, & \text{if } \gamma_{i,k-1} = 1 \\ A_{o,i}x_{i,k-1}, & \text{if } \gamma_{i,k-1} = 0 \end{cases} \quad (5)$$

This mean value $\mu_{i,k}$ encompasses any information about the future evolution of system. That is, conditioned on the past the future evolution depends just on $\mu_{i,k}$ and future decisions. Hence we restrict attention to channel access policies of the form

$$\alpha_{i,k} = \alpha_i(x_{i,k}, \mu_{i,k}) \quad (6)$$

that are functions of the current mean $\mu_{i,k}$ and measurement $x_{i,k}$. Here the mappings $\alpha_i : \mathbb{R}^{n_i} \times \mathbb{R}^{n_i} \rightarrow [0, 1]$ are taken to be time invariant, because the systems dynamics do not change over time. The following example illustrates how the coupling of the plants via the shared wireless channel complicates the selection of appropriate sensor access policies.

Example 2. The wireless interference couples the loops and may lead to instability. Consider for example two identical scalar systems of the form (2) (here $A_{o,i} = 1.1$, $A_{c,i} = 0.4$) with standard normal disturbances. The sensors transmit over a shared wireless channel using a random access mechanism. Suppose that each sensor transmits deterministically ($\alpha_{i,k} = 1$) whenever its state exceeds some threshold (here $|x_{i,k}| > 5$). This choice is justified since sensors may want to avoid transmitting as long as their plants are relatively close to the desirable operating point 0, in an effort to limit packet collisions to other control loops. A simulation is shown in Fig. 2. Initially both plant states are successfully kept within bounds, until at some time both plants exceed the threshold due

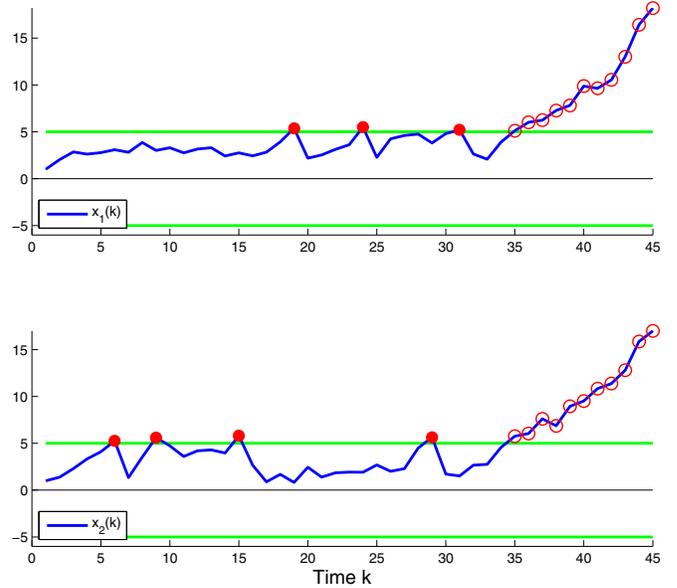


Fig. 2. Coupled evolution of plants over the shared wireless channel in Ex. 2. The evolution of each plant state is plotted, with the time of transmissions shown by closed circles and packet collisions by open circles. Due to the random disturbance both plants happen to exceed their threshold simultaneously at some slot, transmissions collide, plant states grow, and keep on transmitting and colliding. The decentralized random access mechanism results in instability.

to the random disturbance. In that case both sensors transmit and collide. The two plants run in open loop and at the next time step plant states are likely to grow, sensors retransmit, collide again, and so on. \square

Intuitively each sensor should access the channel often enough to close its loop but not too often otherwise the wireless interference might cause loss of stability on the other control loops. We are faced with the following design problem.

Problem: We consider m systems of the form (2) over a shared wireless channel where communication is modeled by (1). We are looking for control-aware random access policies $\alpha_{i,k} \in [0, 1]$ of the form (6) for each sensor $i = 1, \dots, m$ and times $k \geq 0$ so that all control loops are stabilizable. In particular we are interested in mean square stability, i.e., that

$$\limsup_{k \rightarrow \infty} \mathbb{E} [x_{i,k} x_{i,k}^T] < \infty \quad (7)$$

holds for all $i = 1, \dots, m$. Expectation here accounts for the randomness introduced by the disturbances of each plant, randomness of the channel access policies, and randomness of packet successes by (1).

In the following section we present a condition on the sensor access policies so that the produced interference does not violate stability for any system. Then based on this decoupling condition we design appropriate policies in Section IV.

Remark 1. The channel model in (1) can be generalized in several ways. First, even without interference from simultaneous transmissions, a message may not always be successfully decoded at the controller due to noise added to

the transmitted signal or due to wireless fading effects [26]. If successful decoding occurs with some constant probability $q_i \in [0, 1]$, the right hand side in (1) needs to be modified to $q_i \alpha_{i,k} \prod_{j \neq i} (1 - \alpha_{j,k})$.

Second, we can model more complex interference relations that the conservative case in (1) where simultaneous transmissions certainly lead to collisions. Suppose that if a sensor j transmits at the same time slot as sensor i does, a collision occurs on sensor i 's packet with some constant probability $q_c \in [0, 1]$. In other words simultaneously transmitted packets are not always lost. Given that sensor j transmits randomly with probability $\alpha_{j,k}$, the probability that transmission i is affected by sensor j equals the product $\alpha_{j,k} q_c$. The combined effect from all sensors on packet success at link i is modified from (1) to

$$\mathbb{P}(\gamma_{i,k} = 1) = q_i \alpha_{i,k} \prod_{j \neq i} [1 - \alpha_{j,k} q_c]. \quad (8)$$

The results of this paper extend to the general model (8). We can also consider asymmetric models where different sensors interfere differently, but we omit the discussion due to space limitations – see [16] for details. \square

III. STABILIZABILITY UNDER CONTROL-AWARE RANDOM ACCESS

The goal of this section is to derive a condition guaranteeing that the interference due to the random access policies over the shared wireless medium do not violate the stabilizability of any control loop. In particular we consider the following mathematical condition.

Condition 1. There exists a vector $\tilde{\alpha} \in [0, 1]^m$ and positive definite matrices $P_i \in \mathbb{S}_{++}^m$, $i = 1, \dots, m$ such that

$$A_{o,i}^T P_i A_{o,i} - \tilde{\alpha}_i \prod_{j \neq i} (1 - \tilde{\alpha}_j) (A_{o,i}^T P_i A_{o,i} - A_{c,i}^T P_i A_{c,i}) \prec P_i \quad (9)$$

holds for all $i = 1, \dots, m$.

Each linear matrix inequality in (9) depends on the system dynamics described in (2), and there are m such inequalities in the above condition. Our main result is that if this condition holds, then we can select state-based random access policies for each sensor so that interference between loops does not lead to loss of stability. Before we state the result formally, we present an intuitive interpretation of Condition 1.

Suppose that instead of adapting to plant states, each sensor i decides to access the channel with a constant probability $\alpha_{i,k} = \tilde{\alpha}_i \in [0, 1]$ for all time steps $k \geq 0$. Given such a set of policies, the event that a sensor i successfully closes its loop at any time k by (1) becomes an i.i.d. Bernoulli random variable with success

$$\mathbb{P}(\gamma_{i,k} = 1) = \tilde{\alpha}_i \prod_{j \neq i} (1 - \tilde{\alpha}_j). \quad (10)$$

Note that this expression appears explicitly in (9). As a consequence of the i.i.d random transmission success each system in (2) performs random i.i.d. jumps. The interpretation

of condition (9) then is that the functions $x_i^T P_i x_i$ are Lyapunov functions for each system. This is necessary and sufficient for mean square stability, as stated in the following result.

Theorem 1. Consider m systems of the form (2) over a shared wireless channel and random access communication modeled by (1). Suppose each sensor $i = 1, \dots, m$ selects a constant policy of the form $\alpha_{i,k} = \tilde{\alpha}_i$ for all times $k \geq 0$. Then the systems are stable in the mean square sense (7) if and only if there exist positive definite matrices $P_i \in \mathbb{S}_{++}^m$, $i = 1, \dots, m$ such that (9) holds for all $i = 1, \dots, m$.

Proof. For each switched linear system i described by (2), the switching variables $\gamma_{i,k}$ are a sequence of i.i.d. Bernoulli random variables given in (10). Then [27, Cor. 1] states that a necessary and sufficient condition for mean square stability of a random jump linear system is the existence of a positive definite matrix P_i , as in the statement of the theorem, such that

$$\mathbb{P}(\gamma_{i,k} = 1) A_{c,i}^T P_i A_{c,i} + \mathbb{P}(\gamma_{i,k} = 0) A_{o,i}^T P_i A_{o,i} \prec P_i \quad (11)$$

holds. Substituting the expression for the success probability $\mathbb{P}(\gamma_{i,k} = 1)$ given in (10) and rearranging terms we get (9), which completes the proof. \square

We are now ready to state our main result.

Theorem 2. Consider m systems of the form (2) over a shared wireless channel and random access communication modeled by (1). Let Condition 1 hold. If all sensors $j \neq i$ select policies of the form (5)-(6) satisfying

$$\mathbb{E}[\alpha_{j,k} \mid \mu_{j,k}] = \tilde{\alpha}_j, \text{ for all } k \geq 0, \quad (12)$$

then system i is stabilizable, i.e., there exists a policy for sensor i such that system i is mean square stable according to (7). The same holds symmetrically for all $i = 1, \dots, m$.

Proof. Suppose all sensors $j \neq i$ select policies satisfying (12). We will show it suffices for sensor i to select the constant policy

$$\alpha_{i,k} = \alpha_i(x_{i,k}, \mu_{i,k}) = \tilde{\alpha}_i \quad (13)$$

in order to render system i mean square stable. Here $\tilde{\alpha}_i$ is the i th element of the $\tilde{\alpha}$ of Condition 1.

First, note that by comparing (2) and (5) the variables $x_{i,k}$ differ from the variables $\mu_{i,k}$ by a Gaussian quantity, i.e., $\mathbb{E}x_{i,k} x_{i,k}^T = \mathbb{E}\mu_{i,k} \mu_{i,k}^T + W_i$. Hence for mean square stability of $x_{i,k}$ it suffices to prove mean square stability for $\mu_{i,k}$.

Consider the function $V_i(\mu_{i,k}) = \mu_{i,k}^T P_i \mu_{i,k}$, where P_i is the positive definite matrix given in Condition 1. Then consider the value of the function during the evolution of the systems using the specified policy. For conciseness let us denote the values of all $\mu_{j,k}$ for $j = 1, \dots, m$ as $\boldsymbol{\mu}_k$. Conditioned on $\boldsymbol{\mu}_k$ we have by direct substitution from (5) that

$$\mathbb{E}[V_i(\mu_{i,k+1}) \mid \boldsymbol{\mu}_k] = \mathbb{E}[\gamma_{i,k} x_{i,k}^T A_{c,i}^T P_i A_{c,i} x_{i,k} + (1 - \gamma_{i,k}) x_{i,k}^T A_{o,i}^T P_i A_{o,i} x_{i,k} \mid \boldsymbol{\mu}_k] \quad (14)$$

Note that $\gamma_{i,k}$ is a Bernoulli random variable (cf.(1)) with success probability $\alpha_{i,k} \prod_{j \neq i} (1 - \alpha_{j,k})$, where $\alpha_{i,k}$ is constant equal to $\tilde{\alpha}_i$ by our choice. Hence after rearranging terms in (14) we get

$$\begin{aligned} \mathbb{E}[V_i(\mu_{i,k+1}) \mid \boldsymbol{\mu}_k] &= \mathbb{E}[x_{i,k}^T A_{o,i}^T P_i A_{o,i} x_{i,k} + \\ &\tilde{\alpha}_i \prod_{j \neq i} (1 - \alpha_{j,k}) x_{i,k}^T [A_{o,i}^T P_i A_{o,i} - A_{c,i}^T P_i A_{c,i}] x_{i,k} \mid \boldsymbol{\mu}_k] \end{aligned} \quad (15)$$

We then argue that the variables $\alpha_{j,k}$ conditioned on $\boldsymbol{\mu}_k$ are independent of any other variables in the expectation. Note that the policies we consider by (6) for any sensor $j \neq i$ are functions $\alpha_{j,k} = \alpha_j(x_{j,k}, \mu_{j,k})$ of their respective states. Since $x_{j,k}$ equals $\mu_{j,k} + w_{j,k}$ and the disturbances are independent among systems by assumption, we get that the second term in (15) equals

$$\begin{aligned} &\mathbb{E}[\tilde{\alpha}_i \prod_{j \neq i} (1 - \alpha_{j,k}) x_{i,k}^T [A_{o,i}^T P_i A_{o,i} - A_{c,i}^T P_i A_{c,i}] x_{i,k} \mid \boldsymbol{\mu}_k] \\ &= \tilde{\alpha}_i \prod_{j \neq i} (1 - \mathbb{E}[\alpha_{j,k} \mid \boldsymbol{\mu}_k]) \\ &\quad \times \mathbb{E}[x_{i,k}^T [A_{o,i}^T P_i A_{o,i} - A_{c,i}^T P_i A_{c,i}] x_{i,k} \mid \boldsymbol{\mu}_k] \end{aligned} \quad (16)$$

We use the fact that our specific choice of policies satisfy (12), we conclude that (15)-(16) implies

$$\begin{aligned} \mathbb{E}[V_i(\mu_{i,k+1}) \mid \boldsymbol{\mu}_k] &= \mathbb{E}[x_{i,k}^T A_{o,i}^T P_i A_{o,i} x_{i,k} + \\ &\tilde{\alpha}_i \prod_{j \neq i} (1 - \tilde{\alpha}_j) x_{i,k}^T [A_{o,i}^T P_i A_{o,i} - A_{c,i}^T P_i A_{c,i}] x_{i,k} \mid \boldsymbol{\mu}_k] \end{aligned} \quad (17)$$

Finally noting that $x_{i,k} = \mu_{i,k} + w_{i,k}$ and that the variable $w_{i,k}$ is independent $\boldsymbol{\mu}_k$, having zero mean and variance W_i , we can rewrite (17) as

$$\mathbb{E}[V_i(\mu_{i,k+1}) \mid \boldsymbol{\mu}_k] = \mu_{i,k}^T M \mu_{i,k} + Tr(MW_i) \quad (18)$$

where

$$M = A_{o,i}^T P_i A_{o,i} - \tilde{\alpha}_i \prod_{j \neq i} (1 - \tilde{\alpha}_j) [A_{o,i}^T P_i A_{o,i} - A_{c,i}^T P_i A_{c,i}] \quad (19)$$

But note then that Condition 1 states that $M \prec P_i$. In particular, there exists some $\rho \in [0, 1)$ such that $M \preceq \rho P_i$. This result in combination with (18) implies that for any $\mu_{i,k} \in \mathbb{R}^{n_i}$

$$\begin{aligned} \mathbb{E}[V(\mu_{i,k+1}) \mid \boldsymbol{\mu}_k] &\leq \rho \mu_{i,k}^T P_i \mu_{i,k} + Tr(MW_i) \\ &= \rho V_i(\mu_{i,k}) + Tr(MW_i) \end{aligned} \quad (20)$$

Taking expectation at both sides over all involved random variables and repeating over time we finally get that

$$\mathbb{E}[V_i(\mu_{i,k+1})] \leq \rho^k \mu_{i,0}^T P_i \mu_{i,0} + \sum_{\ell=0}^{k-1} \rho^\ell Tr(MW_i). \quad (21)$$

Letting time $k \rightarrow +\infty$, the sum above is bounded due to the fact that $\rho < 1$. This shows that system i is mean square stable

as it satisfies (7). A symmetric argument verifies that this is the case for all systems. \square

This theorem is expressed from system i 's point of view. The important observation is that it prescribes conditions on the policies of all other sensors $j \neq i$ so that they do not interfere with sensor i 's closed loop. Specifically, as long as all sensors $j \neq i$ adhere to the average access rates in (12), then sensor i can still keep its own loop stable. The same holds symmetrically for all loops i .

Flipping the point of view, we can interpret each condition (12) as a requirement that sensor j should satisfy so that it does not interfere with any other closed loop $i \neq j$. If all sensors $j = 1, \dots, m$ adhere to their respective condition, then no interference between closed loops is significant enough to cause anyone's loss of stability. We have thus managed to decouple the wireless interference between closed loops. Moreover we can exploit this decoupling to design each sensor access policy that satisfies the above condition separately across systems. This is described in the following section.

Before we proceed to the design we would like to point out that it is computationally hard to check whether Condition 1 holds for given plant dynamics. Some approaches to address this issue are presented next.

A. Computational issues

For our approach it is necessary to check whether Condition 1 holds. This is computationally hard as it involves a set of m coupled inequalities. Each one is a multi-linear matrix inequality, i.e., a linear matrix inequality with respect to each variable separately if all other variables are fixed. One needs to jointly search for suitable variables $\tilde{\alpha} \in [0, 1]^m$ and $P_i \in \mathbb{S}_{++}^{n_i}$, $i = 1, \dots, m$. We detail some approaches to the problem:

- (i) One approach is to limit the search space by assuming that matrices P_i are fixed. Then one needs to solve for $\tilde{\alpha}$ to ensure that all inequalities in (9) hold. We note here that each inequality i is a linear matrix inequality with respect to the scalar term $\tilde{\alpha}_i \prod_{j \neq i} (1 - \tilde{\alpha}_j)$. As a result, inequality i expresses a convex interval where this term should take values. In particular, for fixed P_i inequality (9) is equivalent to

$$\tilde{\alpha}_i \prod_{j \neq i} (1 - \tilde{\alpha}_j) > c_i \quad (22)$$

for some appropriate constant $c_i > 0$. Taking the logarithm of each side of (22) gives an equivalent convex inequality

$$\log(\tilde{\alpha}_i) + \sum_{j \neq i} \log(1 - \tilde{\alpha}_j) > \log(c_i) \quad (23)$$

with respect to the variables $\tilde{\alpha}$. Feasibility of this equivalent set of convex inequalities for $i = 1, \dots, m$ can be readily checked using available algorithms [28]. In our previous work [16], [17] we have take this approach to explore constant random access policies. For given matrices corresponding to desirable Lyapunov functions we

have characterized the solutions to coupled inequalities of the form (22)-(23).

- (ii) Alternatively if we fix the variables $\tilde{\alpha}$ to some values, checking whether a matrix P_i exists that satisfies (9) is a standard linear matrix inequality problem which is computationally amenable too [28]. Note that here each P_i can be found separately for $i = 1, \dots, m$.
- (iii) Since the search for each set of variables $\tilde{\alpha}$ and P_i separately is tractable, an iterative procedure might be employed. That is, compute one set while the other remains fixed and so on. This is a heuristic procedure and it is uncertain whether it will converge to a feasible point.
- (iv) It is worth noting that in some special cases checking feasibility of Condition 1 is tractable. This happens for example if for each system i the open and closed loop matrices $A_{o,i}, A_{c,i}$ in (2) commute. In that case the matrices are also simultaneously diagonalizable by some matrix $M_i \in \mathbb{S}_{++}^{n_i}$. Performing a state transformation from $x_{i,k}$ to $M_i x_{i,k}$ for each system gives new matrices $A'_{o,i}, A'_{c,i}$ in (2) that are diagonal. To check (9) using these diagonal matrices it suffices to consider P_i to be the identity matrix I . Since P_i is now fixed, we have already argued in (i) above that checking feasibility of (9) is tractable.

IV. DESIGN OF CONTROL-AWARE RANDOM ACCESS POLICIES

After decoupling the interference between the control loops by Theorem 2, in this section we propose suitable sensor access policies adapted to online plant state measurements. In particular we design policies of the form $\alpha_i(x_{i,k}, \mu_{i,k}) \in [0, 1]$ as in (6) that satisfy condition (12), i.e.,

$$\mathbb{E} [\alpha_i(x_{i,k}, \mu_{i,k}) \mid \mu_{i,k}] = \tilde{\alpha}_i. \quad (24)$$

A trivial way of satisfying this condition is to just select a constant random access policy $\alpha_i(\cdot, \cdot) = \tilde{\alpha}_i$ regardless of the current or any past measurements (cf. Theorem 1). Alternatively the sensors could adapt and respond to observed desirable or unexpected events. We present two such types of policies.

A. Innovation-based policy

Consider first *threshold* policies based on the *state innovation*. Before measuring the current plant state $x_{i,k}$, the sensor i knows that the state has a Gaussian distribution with mean $\mu_{i,k}$ given in (5) and variance W_i . Consider then a policy where the sensor does not transmit if the difference or innovation $x_{i,k} - \mu_{i,k}$ between measured and expected state is small, according to a threshold rule of the form

$$\alpha_i(x_{i,k}, \mu_{i,k}) = \begin{cases} 1, & \text{if } \|W_i^{-1/2}(x_{i,k} - \mu_{i,k})\|^2 > t \\ 0, & \text{otherwise} \end{cases}. \quad (25)$$

Here t is a threshold to be designed so that (24) holds, or equivalently

$$\mathbb{P} \left(\|W_i^{-1/2}(x_{i,k} - \mu_{i,k})\|^2 > t \right) = \tilde{\alpha}_i. \quad (26)$$

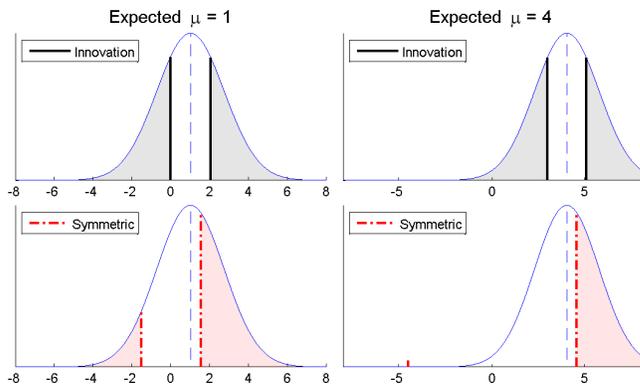


Fig. 3. Illustration of proposed access policies $\alpha_{i,k}$ as functions of the state $x_{i,k}$ for different current mean values $\mu_{i,k}$ (left and right column). The top and bottom row correspond to the innovation-based and symmetric policy respectively. The distribution of $x_{i,k}$, Gaussian with variance W_i is also plotted on the same axis (here $W_i = 3$). The sensor transmits ($\alpha_{i,k} = 1$) when $x_{i,k}$ is in the shaded region in each case. The innovation-based policy is centered around the mean $\mu_{i,k}$, while the symmetric is centered around 0. The probability of transmission (shaded region) is constant equal to $\tilde{\alpha}_i$ in all cases (cf. (26),(29)).

The term $W^{-1/2}$ is included for normalization. Note that the difference $x_{i,k} - \mu_{i,k}$ equals the current system disturbance $w_{i,k}$ (cf. (2), (5)). Since $w_{i,k}$ has a Normal distribution with mean 0 and covariance W_i the random variable $\|W_i^{-1/2} w_{i,k}\|^2$ has a χ^2 distribution with parameter n_i (the size of the vector) [29, Sec.B], denoted here by F_{χ^2, n_i} . Thus the sensor threshold in (26) is chosen as

$$t = F_{\chi^2, n_i}^{-1}(1 - \tilde{\alpha}_i). \quad (27)$$

That is, t corresponds to the $(1 - \tilde{\alpha}_i)$ -quantile of the distribution. This computation is readily available in numerical analysis software. Note that the employed threshold value t is constant, computed during the design phase, and does not need to be updated online.

The implementation of this policy at the sensor is as follows. Upon measuring the current state $x_{i,k}$ the sensor compares it with its expected value $\mu_{i,k}$. If the comparison gives that $\|W_i^{-1/2}(x_{i,k} - \mu_{i,k})\|^2 > t$ the sensor transmits. Then it waits for the acknowledgment of the transmission and updates the new mean value $\mu_{i,k+1}$ according to (5), and the process repeats. See Fig. 3 for a visual illustration of this innovation-based policy.

It is worth noting that the form of our policy resembles the form of Delta-triggered [12], [13] or Lebesgue sampling [30] policies. The works [12], [13] examine control over a shared communication medium, as here, however they focus on simple dynamics (identical and scalar integrators) and further simplify development by approximating packet collisions as i.i.d. packet drops. In contrast we make no approximations and consider general and asymmetric system dynamics. Our adaptive thresholds ensure by design no loss of stabilizability of any control loop due to collisions according to Theorem 2.

B. Symmetric state-based policy

An alternative is to consider policies where the sensor transmits when the plant state $x_{i,k}$ is away from the desirable operating point 0. These policies are centered around 0, i.e.,

$$\alpha_i(x_{i,k}, \mu_{i,k}) = \begin{cases} 1, & \text{if } \|W^{-1/2} x_{i,k}\|^2 > t_k \\ 0, & \text{otherwise} \end{cases} \quad (28)$$

where the term $W^{-1/2}$ is added again for normalization. The threshold is again chosen so that

$$\mathbb{P}\left(\|W^{-1/2} x_{i,k}\|^2 > t_k \mid \mu_{i,k}\right) = \tilde{\alpha}_i. \quad (29)$$

Recall that $x_{i,k}$ is a Gaussian vector with mean $\mu_{i,k}$ and covariance W_i . Thus the variable $\|W^{-1/2} x_{i,k}\|^2$ is a non-central χ^2 random variable with n_i degrees of freedom and noncentrality parameter $\|W^{-1/2} \mu_{i,k}\|^2 \in \mathbb{R}_+$ [29, Sec.B]. This distribution hence depends on the current value $\mu_{i,k}$. The required threshold t_k needs to be chosen as the $(1 - \tilde{\alpha}_i)$ -quantile of the distribution, hence it will change as a function of the dynamically changing mean $\mu_{i,k}$. This quantile computation is readily available in numerical analysis software.

A visual illustration of this symmetric policy is shown in Fig. 3. A difference with the previous innovation-based policy is that the threshold t_k here is not fixed but changes over time in order to satisfy (29). It is also worth noting that the policy we obtain is reminiscent of the event-based control framework of, e.g., [18]–[20], where plant state information is exploited to lower the communication rate. Here in contrast plant state information is used so that no disastrous interference is caused on other control loops over the shared channel. We also point out that unlike the usual event-based rules the thresholds in our case need to be dynamically adapted online at each time step. When a larger or smaller plant state is expected, for example after a packet collision or a successful transmission respectively, the threshold is increased or decreased respectively.

In the following section we present numerical simulations of the proposed policies, where we see that they maintain stability of all control loops and we compare them in terms of control performance. We note that by Theorem 2 our policies are guaranteed by design to not cause wireless interference significant enough to violate the stability of any control loop. It remains however to support theoretically that if all sensors jointly employ the proposed policies in (25) or (28), mean square stability of all loops is guaranteed. We aim to verify this fact in subsequent work.

Remark 2. The exact probability distributions of the plant disturbances $\{w_{i,k}, k \geq 0\}$ are only used in the design of the policies in this section. In particular they are used to select the appropriate thresholds on the plant states in (26), (29). On the other hand if the disturbances have distributions different than Gaussian, Theorem 2 still holds. To derive the appropriate thresholds (26), (29) some inversion of these distributions is required, which might be computationally involved. These cases, as well as the case where the distributions are not known, is a subject of future work. \square

V. NUMERICAL SIMULATIONS

We begin with numerical simulations of the sensor random access policies developed in Section IV to illustrate their ability to mitigate collisions between control loops. For simplicity of exposition we consider two ($m = 2$) identical scalar systems of the form (2) with unstable open loop $A_{o,1} = A_{o,2} = 1.1$, and stable closed loop $A_{c,1} = A_{c,2} = 0.4$ dynamics, as in Ex. 2. We assume both are disturbed by Gaussian noise with covariance $W_1 = W_2 = 2$. Then we have to find values $\tilde{\alpha}_1, \tilde{\alpha}_2$ in Condition 1 which according to Theorem 2 guarantees that the interference between the loops due to packet collisions does not cause instability of any loop. As mentioned in Section III-A for scalar systems checking Condition 1 is computationally tractable. Here we pick $\tilde{\alpha}_1 = \tilde{\alpha}_2 = 0.35$ – other choices are possible too.

We first simulate the innovation-based policy (25). In Fig. 4 we plot the evolution of the two plants as well as the evolution of the innovation-based thresholds selected by each sensor. As can be seen, each sensor transmits whenever its state exceeds the currently selected threshold. We observe that the thresholds track the evolution of the system as they are by design centered around the currently expected value $\mu_{i,k}$ of the state $x_{i,k}$ (cf. (5), (25)). Collisions do occur but not often enough to destabilize the systems. The probability that sensors transmit (equivalently, states exceed threshold) is kept constant by design (cf.(26)).

We then simulate the symmetric policy (28). In Fig. 5 we plot the evolution of the two plants as well as the evolution of the thresholds selected by each sensor. As can be seen, each sensor transmits whenever its state exceeds the currently selected threshold. Unlike the previous innovation-based policy, here the thresholds are by design symmetric around the desirable operating point 0 of the plant state $x_{i,k}$ (cf. (25)). The probability that sensors transmit (equivalently, state exceeds threshold) is kept constant by design (cf. (29)), to mitigate collisions. Specifically we can observe that when a collision occurs, each sensor increases its allowed threshold at the next time slot since a larger state is expected. Similarly when a successful transmission occurs the involved sensor decreases its allowed threshold at the next time slot since state is expected to decrease.

We then perform numerical simulations to evaluate the stability and control performance of the decentralized access mechanisms. We consider a more asymmetric example of systems. Let system 1 be described by the two-dimensional open and closed modes (cf.(2))

$$A_{o,1} = \begin{bmatrix} 0.8 & -1 \\ 0 & 1.1 \end{bmatrix}, A_{c,1} = \begin{bmatrix} 0.5 & -2 \\ 0 & 0.7 \end{bmatrix} \quad (30)$$

Let also the second system have scalar integrator open loop dynamics $A_{o,2} = 1$, and get reset to zero in closed loop $A_{c,2} = 0$. System 1 has an unstable pole at 1.1, hence is more critical than system 2 whose evolution will be stable as long as it gets reset one in a while. Then we need to check Condition 1. After some trials we select a large value $\tilde{\alpha}_1 = 0.75$ for system 1 and a lower one $\tilde{\alpha}_2 = 0.05$ for system 2. For fixed values

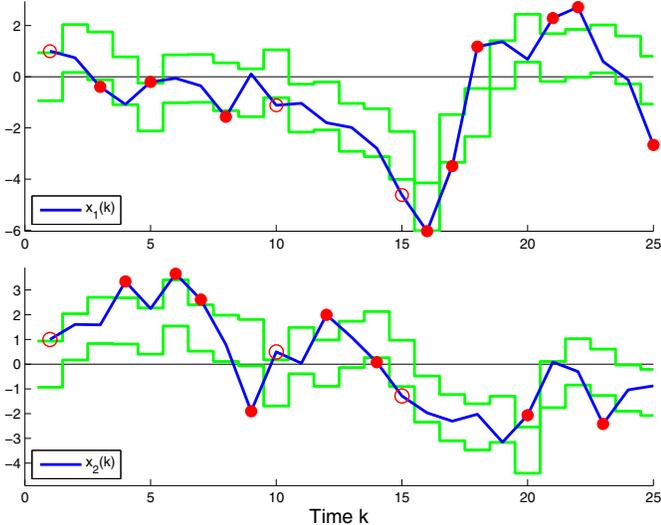


Fig. 4. Simulation of the innovation-based policy (25). The evolution of each plant state and the thresholds are plotted, with the time of transmissions shown by closed circles and packet collisions by open circles. The sensor transmits whenever the state exceeds the thresholds, which are tracking around the evolution of the state.

$\tilde{\alpha}$, as mentioned in Section III-A, we can verify Condition 1 efficiently.

We perform simulations with sensors using three decentralized channel access policies: (i) the constant policy where $\alpha_{i,k} = \tilde{\alpha}_i$ for all sensors $i = 1, \dots, m$ and all times $k \geq 0$ (cf. Theorem 1), (ii) the innovation-based policy in (25), and (iii) the symmetric policy in (28).

In Fig. 6 we plot the average quadratic cost

$$\frac{1}{N} \sum_{k=0}^{N-1} \|x_{i,k}\|^2 \quad (31)$$

of the state evolution for each plant $i = 1, 2$ computed during simulation. We observe that all systems remain stable. We also see that the constant policy which does not take into account online information about the behavior of the systems performs the worst. On the contrary, out of the two online control-aware policies the symmetric one behaves the best, and it actually exhibits a significant performance improvement with respect to the other two. An intuitive interpretation of this fact is that the symmetric policy reacts (i.e., sensor transmits) in a way that tries to keep the plant state close to zero. On the other hand the innovation-based policy reacts only when the incoming noise at some time step is large. This different behavior can also be seen in Figs. 4, 5. We aim to theoretically support this performance advantage of the symmetric policy in future work.

VI. CONCLUSIONS AND FUTURE WORK

We consider the problem of sharing a wireless channel between multiple control loops using a decentralized and easy to implement mechanism. Each sensor randomly accesses the channel however packet collisions arise in the shared

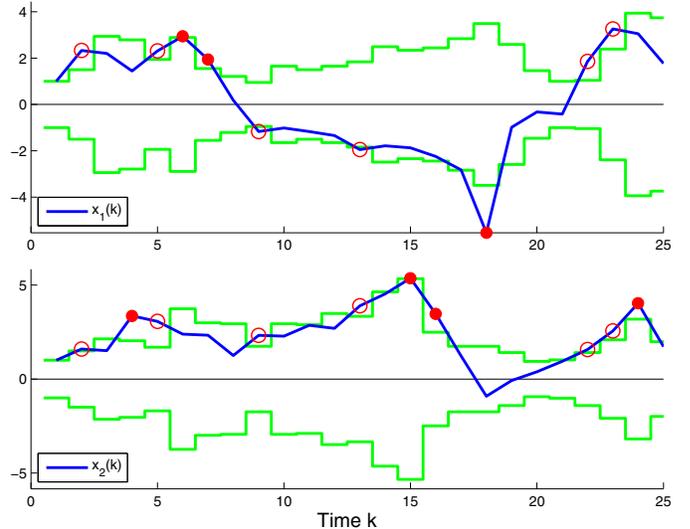


Fig. 5. Simulation of the symmetric policy (28). The evolution of each plant state and the thresholds are plotted, with the time of transmissions shown by closed circles and packet collisions by open circles. The sensor transmits whenever the state exceeds the thresholds, which are by design symmetric around 0. When a transmission succeeds the corresponding threshold is decreased at the next slot, while when packets collide both thresholds are increased.

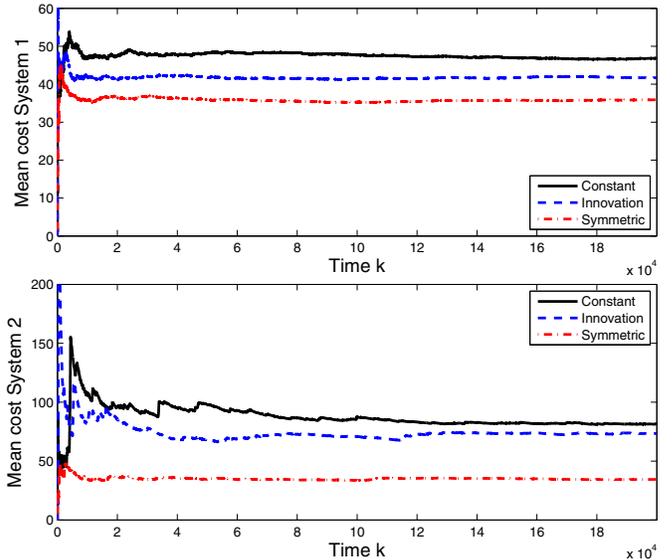


Fig. 6. Comparisons of control performance using three decentralized sensor access policies: (i) the constant one that does not take into account online plant information, (ii) the innovation-based one transmitting when plant behaves much differently than expected, and (iii) the symmetric one transmitting when the norm of the plant states exceeds some threshold. All lead to stability, and the symmetric one has significantly better control performance compared to the others.

medium due to the uncoordinated transmissions. Our goal is to guarantee that this wireless interference does not violate stability of any of the control loops. Towards this goal we derive a sufficient mathematical condition for the sensor access policies. This condition decouples the systems and is employed

in order to design control-aware sensor access policies adapting online to plant evolutions.

The problem of developing computationally efficient algorithms for verifying our decoupling condition for control over a shared wireless channel (Condition 1) requires further exploration. The stability of the plants under the two proposed access mechanisms, which here is only shown in numerical simulations, requires theoretical support. Apart from stability it is important to design decentralized channel access mechanisms with control performance guarantees, as well as guarantees about their resource utilization, e.g., how much transmit power is consumed by the wireless sensors [26]. Future work also includes the design of sensor access policies when sensors do not measure the plant state directly but rely just on noisy plant outputs.

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